The Rich and the Rest

J. Schulz
M. Milaković

September 23, 2019

Working Paper

Abstract

The last decades have seen a surge in measured top wealth inequality across industrialized countries. The canonical random growth model with homogeneous return distributions cannot explain the speed of this increase in aggregate top inequality, since it generates too slow transitions in response to shocks. In contrast to that, recent theoretical contributions attribute this rise to differential expected returns to net wealth resulting from individual differences in innate ability (type dependence) or wealth (scale dependence) (Gabaix et al., 2016). However, this result on the theoretical possibility of differential returns coupled with aggregate evidence is still in need of microlevel confirmation. We propose a parsimonious test for scale-dependence that only needs wealth microdata and apply it to a newly comprised data set on the richest Germans from a rich list in combination with survey data. Here we show that the data at hand indeed suggests scale-dependent wealth accumulation for the richest Germans. However, in contrast to this "Data First" interpretation, we also show that this finding might be fully explained within plausible parameter estimates by differential biases affecting the survey study and rich list, primarily resulting from the equiprobable sampling used for the survey. We thus demonstrate that the pre-analytic vision might heavily influence the inferences drawn from the very same dataset, where the different interpretations imply estimates for the total wealth of the richest that can differ by more than one order of magnitude. Our results show that aggregate findings within scale-free systems such as the wealth distribution of the richest might be spurious, when the sampling method does not properly take their large degree of heterogeneity into account.

1 Introduction

The last decades have seen a surge in measured top wealth inequality across industrialized countries. The canonical random growth model with homogeneous return distributions cannot explain the speed of this increase in aggregate top inequality, since it generates too slow transitions in response to shocks. In contrast to that, recent theoretical contributions attribute this rise to differential expected returns to net wealth resulting from individual differences in innate ability (type dependence) or wealth (scale dependence) (Gabaix et al., 2016). However, this result on the theoretical possibility of differential returns coupled with aggregate evidence is still in need of microlevel confirmation. We propose a parsimonious test for scale-dependence that only needs wealth microdata and apply it to a newly comprised data set on the richest Germans from a rich list in combination with survey data. Here we show that the data at hand indeed suggests scale-dependent wealth accumulation for the richest Germans. However, in contrast to this "Data First" interpretation, we also show that this finding might be fully explained within plausible parameter estimates by differential biases affecting the survey study and rich list, primarily resulting from the equiprobable sampling used for the survey. We thus demonstrate that the pre-analytic vision might heavily influence the inferences drawn from the very same dataset, where the different interpretations imply estimates for the total wealth of the richest that can differ by more than one order of magnitude. Our results show that aggregate findings within scale-free systems such as the wealth distribution of the richest might be spurious, when the sampling method does not properly take their large degree of heterogeneity into account.

For the upper tail of empirical wealth distributions, there exists a wide-ranging consensus that it can be described by a power law or Pareto-type distribution. Table 7 in Appendix B summarizes some of the major studies on this distributional structure for the richest for several countries and time periods. Judging from the evidence collected there, this property of empirical wealth distribution seems very robust as it holds even in medieval Hungary (Hegyi et al., 2007) or in ancient Egypt (Abul-Magd, 2002) and with respect to different proxies for wealth. Power law distributions are seemingly spatially and temporally ubiquitous. Power law tails might be found in the Western world, like in Austria, Canada, Germany, Sweden, the UK and the US (Bach et al., 2011; Brzezinski, 2014; Castaldi and Milaković, 2007; Coelho et al., 2005; Cowell, 2011; Drăgulescu and Yakovenko, 2001; Eckerstorfer et al., 2016; Levy, 1998, 2003; Levy and Solomon, 1997), but they also hold for less developed countries like China, Russia and India (Brzezinski, 2014; Ning and You-Gui, 2007; Sinha, 2006) and across varying time-frames.

Gabaix (2009) and Luttmer (2010) review various generating mechanisms for power laws. Given its empirical universality, any generating mechanism for the emergence of such a power law should also be based on a property that is common across the various considered time-periods, countries and proxies for wealth. One such property that is common at least across the different varieties of capitalism is the type of assets that the richest individuals hold: assets that are continuously reinvested into the same asset class, that is, speculative real-estate, stocks and other financial assets (Davies and Shorrocks, 2000;

Wachter and Yogo, 2010).[1] Thus, a random growth model featuring a multiplicative component seems to be the most adequate candidate generating mechanism. The theory of stochastic multiplicative processes to explain the emergence of power law tails traces back to a long history (Champernowne, 1953; Simon, 1955; Wold and Whittle, 1957; Simon and Bonini, 1958; Mandelbrot, 1960; Steindl, 1965; Kesten, 1973; Milaković, 2003; Castaldi and Milaković, 2007) in reduced-form processes which, however, lack economic microfoundations. This generating mechanism has recently gained traction again within more economically motivated models of a partial and general equilibrium flavour which endogenously generate power law tails in wealth distributions from stochastic capital or asset accumulation (Levy, 2003; Levy and Levy, 2003; Nirei, 2009; Benhabib et al., 2011; Toda, 2014; Piketty and Zucman, 2015; Hubmer et al., 2016; Aoki and Nirei, 2016; Benhabib and Bisin, 2018).

The literature on random multiplicative growth has typically placed rather weak assumptions on the return distributions governing the stochastic process. One important exception is the assumption of a homogeneous return distribution or, put differently, an equilibrating tendency for the expected, risk-adjusted rate of return. This is consistent with the implications of (semi-strong) informationally efficient capital markets and the notion that investor's superior talent in either fundamental or technical analysis cannot lead to excess returns over extended time-frames (Fama, 1965, 1970, 1991). Indeed, as Levy (2003) and Levy and Levy (2003) show experimentally and through the use of Monte Carlo simulations, the scope for differential talent is very limited in light of the Pareto distribution in wealth: If one group of investors consistently outperforms another group of less talented investors by only a tiny margin in terms of their expected returns,[2] the functional form of the emergent stationary distribution differs significantly from a Pareto type and is concave on a double-logarithmic scale. We dub this hypothesis of homogeneous return distributions "Theory First".

In contrast to that, a much more recent strand of literature has challenged the homogeneity hypothesis on theoretical and empirical grounds. Bach et al. (2017) and Fagereng et al. (2018) find excess risk-adjusted returns to wealth portfolios for the richest individuals with the latter finding also evidence for persistence in abnormal returns indicating persistent heterogeneity in financial information and talent. From a more theoretical perspective, Luttmer (2011) and Gabaix et al. (2016) build on the well-known flaw of random growth models that they typically generate very slow transitions. The former puts this in terms of the stationary distribution of assets with a much too high half-life of assets, while the latter argues equivalently in terms of the rate of convergence to the new stationary distribution

[1] From an accounting standpoint, this continuous reinvesting is closely related to saving. Again, there is a consensus in the literature that propensities to save are strongly positively correlated with (lifetime) income or wealth (Dynan et al., 2004; Jappelli and Pistaferri, 2014). This also holds for entrepeneurial households (Quadrini, 1999). As a major reason for this relationship, Deaton (2003) identifies credit constraints that are only binding for low wealth households and individuals.

[2] They consider two Gaussian return distributions only differing in their expected value. If this expected return only differs by one percentage-point or more, the stationary distribution is significantly different from the Pareto type.

in response to a parameter shock that is much too slow to account for the empirically observed rise in top-level income inequality. Gabaix et al. (2016) and Jones and Kim (2018) thus put forward the hypothesis of heterogeneous returns to explain the observed rise in income and wealth inequality, where excess returns either are correlated with wealth levels ("scale dependence") or result from differential talent ("type dependence"). Within (semi-strong) informationally efficient capital markets, scale-dependence can only occur when the set of investment opportunities increases in wealth. The case of hedge funds and some private banks seems to provide at least anecdotal evidence for this as hedge funds typically require high minimum investment inlays (King and Maier, 2010), while some private banks like JP Morgan Chase's require their clients to be two-digit millionaires and hold at least 10 million USD in investible assets (Glazer, 2016). With respect to type dependence, Gabaix et al. (2016) circumvent the aforementioned problem identified in Levy (2003) and Levy and Levy (2003) that differential talent is inconsistent with a Pareto distribution by assuming that "high growth types" only stay in the high growth regime for a limited amount of time and cannot return there. While this idea is theoretically very appealing, it introduces another degree of freedom into any empirical investigation, that is, the number of periods abnormal returns have to prevail for type-dependence to exist. In addition to that, given our limited dataset without information on investor's sophistication, as discussed in Chapter 3, this notion of type-dependence is phenomenologically equivalent to scale-dependence, since we cannot control for investor's ability. Put differently, with our dataset, we cannot empirically distinguish between the hypothesis that individuals are rich because of their excess returns or the alternative hypothesis that they have excess returns because they are rich. We will thus only focus on testing for scale dependence and label this hypothesis "Data First", as it was historically driven by puzzles posed by the observed behaviour of top incomes in relation to the standard random growth model.

The remainder of this paper is organized as follows: Chapter 2 introduces a simple model in the form of a stochastic partial differential equation that allows inferences on the scale-dependence in the accumulation process from the observed stationary distribution. Chapter 3 introduces the two samples and discusses our estimation procedure. The following Chapter 4 presents our results and puts forward two mutually exclusive but plausible explanations for the observed behaviour in the data. Chapter 5 concludes and discusses implications of our results for further research.

2 Model

To inform our empirical investigation, we consider a parsimonious, reduced-form random growth model introduced by Gabaix (1999). In particular, this reduced-form allows to examine the role of scale-dependence for the tail exponent of the stationary distribution. He shows that there are essentially two ingredients to generate power law distributions: first, stochastic random growth and second, a reflecting boundary as a stabilizing force for the process. It is a well known result that stochastic growth models need some kind of stabilizing force to account for the empirically established power law behaviour in the top tail of wealth distributions. Stabilizing forces that were proposed in the literature are reflecting boundaries, entry/exit-mechanisms and mean-reverting behaviour (Gabaix, 2009). For simplicity, we opt for the reflecting boundary. The basic mechanism boils down to the statement that if wealth grows *multiplicatively* by stochastic factors and all wealth levels are bounded away from a finite valued non-negative *reflecting boundary*, a power law as the stationary distribution emerges from the process. Consider the Markov diffusion with support over the real half-line $(0, \infty)$ for the wealth w of a typical, infinitely-lived household or individual normalized by total wealth given by

$$dw_t/w_t = \mu(w)dt + \sigma(w)dB_t, \tag{1}$$

where μ is the mean growth rate of normalized wealth, σ its standard deviation and dB_t are Wiener increments. The reflecting boundary is introduced in the sense that for all individuals with $w_i > w_{min}$ for a small w_{min}, equation (1) above applies, whereas all $w_i \leq w_{min}$ are set to w_{min} for all individuals i. As Gabaix (1999) shows and we rederive in more detail in Appendix A, the stationary distribution of the right tail is that of a power law, with the tail exponent α given by

$$\alpha(\gamma, \sigma) = 1 - 2\frac{\gamma(w) - \bar{\gamma}}{\sigma^2(w)} + \frac{w}{\sigma^2(w)}\frac{\partial\sigma^2(w)}{\partial w}, \tag{2}$$

where $\bar{\gamma}$ is the average wealth growth rate and $\gamma(w)$ the (normalized) mean growth rate for a given wealth level w. Expression (2) has intriguing and intuitive comparative statics with respect to the degree of scale-dependence in both mean growth rates γ and variance σ^2. Whenever the expected (excess) mean growth rate $\gamma(w) - \bar{\gamma}$ increases in wealth levels, that is, positive scale-dependence with respect to average growth, the tail exponent decreases and inequality goes up. In contrast to that, when variance exhibits positive scale-dependence, $\partial\sigma^2(w)/\partial w > 0$, tail exponents tend up and system-wide inequality decreases.

In particular, Zipf's (1949) law with $\alpha = 1$ is an interesting limit case for a situation without any scale-dependence of neither a positive nor a negative type, that is, $\gamma(w) = \bar{\gamma}, \forall w \in \mathbb{R}^+$, and $\partial\sigma^2(w)/\partial w = 0$. These two conditions are typically called Gibrat's law after the seminal study by Gibrat (1931). As can be seen, in the given set-up, Gibrat's law in growth rates is a sufficient condition for Zipf's law to hold in wealth levels. Córdoba (2008a,b) proves that it is also a necessary condition. In our empirical analysis, we take the Zipfian benchmark in the stationary distribution as an indication for scale-independence and statistically significant deviations as evidence for the contrary. We also want to emphasize another theorem in Gabaix (1999), namely, that the ensemble of distinct sets of wealth levels each characterized by Zipf's law can also be characterized by Zipf's law. Thus, finding a tail exponent of unity for different subsets of the upper tail of a particular wealth distribution would indicate that the whole tail is characterized by the same power law with this tail exponent of unity. By the argument above, this implies that if the random growth process within each subset of the population is independent of scale, then the random growth process for the whole population also has to be scale-independent.

3 Data and Estimation

To test the hypothesis of scale-dependence, we examine the properties of two samples for the upper tail of the German wealth distribution on two distinct scales. In particular, for our methodology, we need two samples that exhibit power law right tails, whose covered ranges do not overlap. We find these conditions fulfilled in two very different kinds of samples, a rich list and a survey study.

3.1 The SOEP Sample

The *Socio-Economic Panel* (SOEP) conducted by the *Deutsches Institut für Wirtschaftsforschung* is probably the most prominent source for German microdata. The 2002, 2007 and 2012 waves of this panel also include an item on personal and household wealth which will be used in our analysis. Utilizing different weighing and imputation techniques for the market value and for disaggregation to the individual value, the *SOEP* sample claims to be representative of the whole German population - implying that each person or household in Germany is chosen with equal probability (Frick et al., 2007). Assuming $82,500,000$ to be the total number of individuals in Germany, the sampling ratio, therefore, is about 0.035 per-cent (Statistisches Bundesamt, 2017).

These three waves, however, do not include a single specific item for total personal wealth, but rather for different asset classes. The total wealth used in the subsequent chapters is therefore calculated as the sum of the value of financial assets (item *PLC0329*), the value of property (item *PLC0357*), the value of commercial enterprises (item *PLC0366*) and the value of tangible assets (item *PLC0371*) held by an individual, substracting the value of debt from private individual credit (item *PLC0422*). The fact that the *SOEP* provides named data (while obviously anonymized) also makes it possible to investigate the time development of assets, in particular in terms of mobility and growth rates in the value of wealth portfolios. Non-respondents for a particular asset class were completely taken out of the sample which could be a source of bias. As these non-respondents typically, however, were only non-respondent for a single asset-class, including them would probably be a greater source of bias which is why this alternative was chosen. Furthermore, all values are inflation-adjusted which implies that they are comparable between the sample periods. The maximum, inflation-adjusted wealth level across all periods is about 70 million €, far from the minimum wealth level in the *manager magazin* (mm) rich list we also consider.

3.2 The manager magazin Rich List

While the *SOEP* sample might provide a reasonable approximation to the actual distribution of wealth for the majority of German individuals, wealth data from household surveys becomes more inaccurate for the tails of the distribution (Davies and Shorrocks, 2000). Casual empiricism indeed suggests that the 70 million € as the maximum wealth level in the *SOEP* is far from being “representative” for the richest individual in Germany. To close this gap and to provide a second, non-overlapping sample on another scale, we consider a rich list for the richest German individuals. While having many drawbacks, the use of rich lists is well established in the literature, particularly for countries like Germany, where tax data does not exist or is not publicly available due to privacy regulations. The German journal *manager magazin* provides a list of the 500 richest Germans for the years $2010 - 2016$. The sample does not overlap with the survey data, as the minimum wealth level reported for all years for the rich list is about double the maximum wealth level in the survey.

The data itself, however, suffers from several issues, such as not consistently distinguishing between household, family and individual wealth. Furthermore, the data for 2012 is missing completely which proves problematic especially for investigations of growth rates that necessitate the comparison of two subsequent sample periods. Another important problem is that the *manager magazin* data only reports the wealth data on a two decimal digit level in billions. In particular, the data likely exhibits so called heaping effects or "digit-preference", as the data seems abnormaly clustered at increments of 50 million € (Heitjan and Rubin, 1991; Schneeweiß et al., 2010). Since we cannot assume that this "digit-preference" is uniformly distributed across the reported data,[3] the heaping errors will in general not cancel out and thus, lead to biased estimators. We will discuss this issue in more detail in the "Theory First" section on the interpretation of results. Finally, the *manager magazin* staff did not disclose any information on the detailed data collection procedure. In personal correspondence, they only stated that the reported wealth levels are based on data available in official archives, from lawyers and asset managers as well as the respective individuals themselves. Thus, it proves difficult to judge the reliability of the provided data. Notwithstanding all these limitations, the *manager magazin* sample is the only available source for named data in this very high wealth region for Germany and therefore seemingly the only way to get a fuller grasp on the German wealth distribution in question. To bring the *SOEP* and the *manager magazin* data together, it is assumed that the *manager magazin* rich lists report personal wealth only. Judging from the actual lists, this seems to be the case for the vast majority of given wealth levels, as, for most of the cases, only a single name is reported for a particular wealth observation. Whenever the respective wealth shares in a family or household are reported or publicly available, the data is also disaggregated on a personal level for each individual in this household or family.[4]

3.3 Estimation

Our empirical analysis is primarily based on the parameter estimates of the power law distributions in the upper tail of the *SOEP* and the *manager magazin* sample. We interpret these as the stationary distributions resulting from a general random-growth process as described in equation (1). The assumption that the empirically observed state coincides with the stationary state of the distribution for time $t \to \infty$ is frequently challenged, though. Especially Gabaix et al. (2016) and Luttmer (2011) show that the convergence to

[3] In fact, the largest wealth levels seem typically reported with much higher accuracy.

[4] All rich lists by the *manager magazin* were digitalized manually and are available upon request.

a new stationary distribution from a shock e.g. to the variance of the underlying random growth process is extremely slow. Given slow convergence, it is questionable whether the empirical distribution truly reflects the dynamics of an underlying random growth process or if it is merely in a transient state to the new asymptotic state. However, as Levy and Levy (2003) show, while convergence to the *asymptotic* distribution is indeed very slow for these types of random growth processes, the convergence to the *approximate* power law is much faster. Thus, inference from this distribution approximating the stationary state seems indeed possible to the underlying dynamics of the data-generating process.

The estimation of the tail exponent of the power law is done by *Maximum Likelihood Estimation* (MLE). Clauset et al. (2009) show that an MLE fit is the least biased method to estimate the characteristic exponent for power laws, compared to OLS methods or fitting a linear function onto the power law on a double-logarithmic scale.[5] Even though we estimate from a discrete dataset, we estimate the power law for its continuous analogue, as the analytical results for differential biases in the results section are based on the continuous version.[6] Our results are not materially sensitive to the choice between the discrete and continuous estimator. Standard errors are determined from the Gaussianity of the MLE, as shown by De Haan and Resnick (1997). For the determination of the minimum wealth level w_{min} from which on the power law applies in the *SOEP* sample, we use the method by Clauset et al. (2009) which is the standard procedure in the field. There, it is also shown that this method is outperforming other possible procedures, such as minimizing the Bayesian Information Criterion (BIC). It is based on the goodness-of-fit of a power law distribution for an increasing sample-size starting including decreasing wealth levels from the maximum onwards (reverse order statistic) according to the Kolmogorov-Smirnov (KS) test. According to this procedure, these levels are reached for $w_{min} = 280,000$ for the 2002 sample, $w_{min} = 200,000$ for the 2007 period and $w_{min} = 180,000$ for the 2012 sample. All estimates are plausible, as it seems reasonable that from about a wealth of 200,000 € onwards, exclusively or at least primarily multiplicative returns apply.[7] Since the rich list should, by our theoretical assumptions, be completely characterized by a power law, we do not estimate w_{min} but rather take it directly from the data. Therefore, w_{min} is chosen as the minimal observed observation for each period. This is also advantageous in another sense, as these w_{min} levels ensure that the power laws always span at least two orders of magnitude. Two orders of magnitude are usually considered to be a minimum requirement for a power law to be present at all (Stumpf and Porter, 2012). The only adjustment made was to neglect all of the lowest wealth observations for which there existed less than three observations to avoid any bias by the truncation of the sample to only 500 individuals.

[5] Cf. also Goldstein et al. (2004) for a more rigorous analysis of different graphical methods and their respective shortcomings compared to an MLE.

[6] The MLE is introduced and discussed in Appendix H in more detail.

[7] The lower tail of the wealth distribution is well approximated by an exponential type, in particular the Gamma distribution. The Gamma distribution also emerges as the combinatorially most likely or entropy-maximizing distribution, when both additive and multiplicative processes together (an arithmetic mean constraint and a constraint on the logarithmic mean) are assumed (Milaković, 2003). The minimum here can, therefore, also be interpreted as the threshold level after which the growth process of wealth is multiplicative according to equation (1). Material available upon request.

Finally, with respect to the growth rates, we would expect them to follow a particular distribution, the *Laplace distribution*. This expectation is purely based on the fact that we approximate wealth growth by the logarithmic difference in wealth levels, that is, $g_{i,t} = log(w_{i,t}) - log(w_{i,t-1})$ as the growth rate of individual i between periods t and $t-1$. It can be shown that $log(w)$ follows an exponential distribution if w follows a power law and the difference between two exponentially distributed variables is Laplacian (Kotz et al., 2001). This (possibly asymmetric) Laplacian distribution of returns r has a probability density function given by

$$f(r, m, \sigma_l, \sigma_r) = \begin{cases} = \frac{1}{A} e^{-|\frac{r-m}{\sigma_l}|} \text{ for } r < m & \text{(3a)} \\ = \frac{1}{A} e^{-|\frac{r-m}{\sigma_r}|} \text{ for } r \geq m, & \text{(3b)} \end{cases}$$

where m is a location parameter with $m > 0$, $\sigma_l, \sigma_r > 0$ denote the scale parameters for the region greater or smaller than m and with $A = \sigma_r + \sigma_l$ to normalize the distribution.The symmetric Laplacian is a special case of the asymmetric Laplacian in equations (3a) and (3b) with $\sigma = \sigma_l = \sigma_r$ resulting in a Probability Density Function (PDF) given by

$$f(r, m, \sigma) = \frac{1}{2\sigma} e^{-|\frac{r-m}{\sigma}|}.$$

4 Results

4.1 Distributional Results

For both the *SOEP* as well as the *manager magazin* samples, the empirical Complementary Cumulative Distribution Functions (CCDFs) after the estimated thresholds w_{min} seem to be approximately linear on a double-logarithmic scale, indicating indeed power law behaviour for the richest individuals within the German wealth distribution in both samples (cf. Appendices C and D). The parameter estimations for the power law region within the *SOEP* are given in Table 1.

SOEP	2002	2007	2012
$\hat{\alpha}$	1.3144	1.0978	1.2982
S.E.	0.0423	0.0324	0.0354
$\hat{w}_{min}$	280,000	200,000	180,000
w_{max}	70,550,000	30,600,000	16,000,000
N	961	1,260	1,332

Table 1. Summary of estimates for the power law region of the *SOEP*. w_{min} and w_{max} in € deflated with index year 2010. Minimum determined by the method of Clauset et al. (2009). α estimates by MLE, standard errors for the α estimate determined by Gaussianity of the Hill estimator (DeHaan and Resnick, 1997).

Two peculiarities stand out here: First, the parameter estimates for the characteristic exponent together with the associated standard errors indicate that Zipf's law with $\alpha = 1$

can be rejected for all sample periods at 95% confidence. Secondly, the maximum wealth level for all years is implausibly low and not anywhere near the minimum wealth levels reported in the manager magazin. The former result will be the foundation for our "Data First" interpretation in the subsequent subchapter, the latter is evidence for our "Theory First" interpretation in the subchapter after.

Consider now the estimates for the *manager magazin* sample in Table 2. As can be clearly seen, the $\hat{\alpha}$ MLEs are for all years very close to the Zipfian benchmark, except for 2016. This, however, is also the only year for which wealth held in foundations or charity organizations was included in the estimated wealth levels, according to the *manager magazin* staff. Thus, the tail exponents for the differing years do not exactly measure the same quantity and are thus not comparable which is why we mostly discard 2016 in our analysis. Note that the inequality for the whole sample period seems to be very high compared to other studies who typically estimate characteristic exponents above unity. Indeed, for all studies described there that are concerned with contemporary Western countries, the estimated characteristic exponent is much higher. Only the cases of contemporary Russia and India (Brzezinski, 2014; Sinha, 2006) and of medieval Hungary (Hegyi et al., 2007) show comparable degrees of inequality in their samples. It is noteworthy that such a result contradicts the conventional wisdom that wealth in Germany is more equally distributed than, for example, in the UK or the US.[8] This huge degree of inequality might be partially explainable by the large relevance of intergenerational wealth transmission in Germany. Indeed, casual empiricism suggests that the *manager magazin* sample lists primarily persons whose families trace back to a long dynastic history, for example the Quandt family with Susanna Klatten as its member and richest woman in all sample periods. Also, about 70% of large and old German corporations are still controlled by the owning families (Bergfeld and Weber, 2011). Therefore, the conjecture seems justified that the high degree of inequality in the "richest club" for Germany is partially attributable to its high relevance of dynastic wealth accumulation, especially for the rich.

mm	2010	2011	2013	2014	2015	2016
$\hat{\alpha}$	0.9983	0.9863	1.0999	0.9874	0.9358	0.7615
S.E.	0.00447	0.0442	0.0495	0.0443	0.0419	0.0341
$\hat{w}_{min}$	0.2	0.2	0.25	0.25	0.25	0.2
w_{max}	17.1	19	23.95	31	26.5	30
N	499	498	494	497	500	500

Table 2. Summary of estimates for the manager magazin. w_{min} and w_{max} in billion € deflated with index year 2010. α estimates by MLE, standard errors for the α estimate determined by Gaussianity of the Hill estimator (De Haan and Resnick, 1997).

Since our primary goal is to examine possible scale-dependence in the random growth process, we also look at the growth rate distributions in more detail. To make results comparable between the *manager magazin* and the *SOEP*, we construct the growth rates

[8] Cf. e.g. Milanovic (2016) for reference.

for a 5 year increment. Several non-parametric tests reject the null hypothesis of distributional equivalence between the two samples for both considered periods but fail to reject it within the sample between periods (cf. Appendix E). Thus, the (tadmittedly limited) evidence seems to suggest that the growth process is time-invariant but scale-dependent between samples. The expected parametric distribution, the Laplacian, also is seemingly approximating the data very well.[9] This is indicated by the fact that the growth rate distributions display the characteristic "tent-shape" on a semi-logarithmic scale which can be seen in Figures 1 to 4. Notice that the presence of non-Gaussian growth rate distributions alone indicates the fact that the growth process is not independent in time which would induce, by the Central Limit Theorem, Gaussian growth rates. In fact, of course, stochastically multiplicative growth processes like the one in equation (1) responsible for the emergence of the power law in levels are path-dependent and thus violate independence.

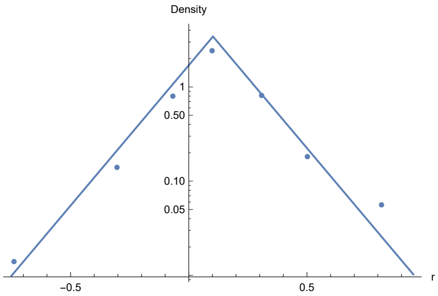

Figure 1. *Growth Rates mm 2010/15.*

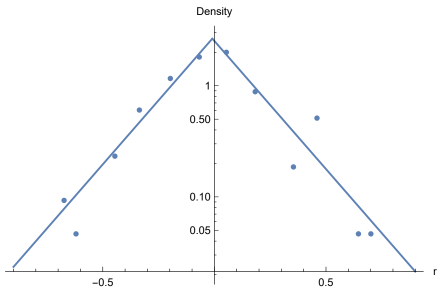

Figure 3. *Growth Rates SOEP 2002/07.*

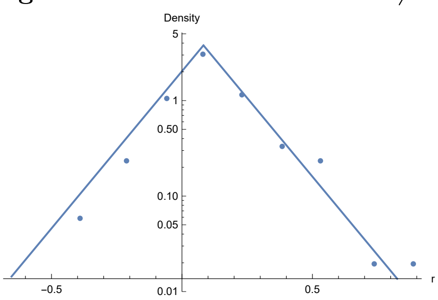

Figure 2. *Growth Rates mm 2011/16.*

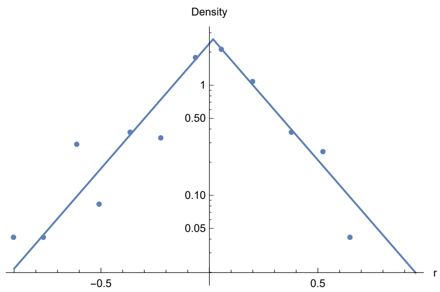

Figure 4. *Growth Rates SOEP 2007/12.*

Note: 12 bins each. Fits by MLE.

The parameter estimates for the Laplacian distribution strengthen the impression by the non-parametric tests. The estimates for m and σ do not vary too much within the respective samples but are vastly different between the considered sample types. While the location parameter m is not significantly different from zero in the *SOEP* sample, zero is not included into the 95% confidence interval for both considered periods which implies that for both periods, m is significantly higher than zero on a 5% significance level. Thus,

[9]The standard procedure in the field to test for the presence of a Laplacian distribution is to fit a *Subbotin* or *Exponential Power Distribution* to the data (Subbotin, 1923). Since the Subbotin distribution includes the Laplacian as a special case for the shape parameter $\kappa = 1$, this fit provides a convenient test. As we show in Appendix F, a shape parameter of unity cannot be rejected for all considered distributions.

the average person or family that was a member of the ´´richest club" in Germany for these two periods benefited in terms of absolute wealth. Also, the dispersion parameter σ is significantly lower than the analogue in the power law regime of both considered intervals in the *SOEP* sample.

Estimated Parameter	$\hat{m}$	$\hat{\sigma}$
manager magazin Sample		
2010/15	0.102355	0.145786
S.E. (2010/15)	0.00904839	0.0095956
2011/16	0.0824254	0.131932
S.E. (2011/16)	0.0116694	0.00819033
SOEP		
2002/07	0.0280287	0.370958
S.E. (2002/07)	0.0144158	0.00768654
2007/12	0.0214543	0.374504
S.E. (2007/12)	0.0148786	0.00755357

Table 3. Summary of estimates for the Laplacian distribution for power law regime of the *SOEP* sample and the *mm* sample.

4.2 Data First

Taking the data and estimation results at face value would imply limited evidence for the existence of scale-dependence within the wealth accumulation process. If we assume the process (1) to hold, an upwards deviations from the Zipfian benchmark for the stationary distribution implies that at least one of either negative scale dependence of the mean growth rate γ or positive scale-dependence of the risk-term $\partial\sigma^2(w)/\partial w > 0$ has to exist. Indeed, this seems to be the case for the within-sample growth process of the survey study but not the one governing the rich list, as in the former, the tail exponent is significantly higher than the Zipfian benchmark which, however, cannot be rejected in the latter. Approaching scale-dependence within-sample from the estimate for the tail exponent bears the significant advantage that the analysis is not based on a partition of the empirical growth rate distributions in wealth levels that is ultimately arbitrary to compare mean and variance of returns.

Notice that both types of growth processes - the scale-dependent one for the relatively lower wealth levels and the scale-independent one for the highest wealth levels - are entirely consistent with the conventional wisdom on the existence of a trade-off between risk and return. The rationale for this tradeoff is simply that competitive financial markets should lead to a (tendency for the) equalization for risk-adjusted returns (Milaković, 2003). For example, the canonical intertemporal capital asset pricing model by Merton (1973) for stock markets suggests that the conditional expected excess return should grow linearly with its conditional variance.[10] This is obvious for the highest wealth levels, where, by the above argument, both the first and the second moment of the expected growth rate distribution resulting from the stochastic growth process are independent of scale and thus, the risk-preferences seem to be approximately identical for all individuals within this group. For the *SOEP* power law regime, on the other hand, a risk-return trade off implying both $\partial\gamma(w)/\partial w > 0$ and $\partial\sigma^2(w)/\partial w > 0$, is still consistent with $\hat{\alpha} > 1$. For an upwards deviation with respect to the Zipfian benchmark like in the parameter estimates, it only has to hold that the positive scale-dependent effect with respect to the variance has to dominate the positive scale-dependence with respect to the expected value. More formally, assuming the risk-return tradeoff to hold, $\alpha > 1$ implies

$$\frac{w}{\sigma^2(w)}\frac{\partial\sigma^2(w)}{\partial w} > 2\frac{\gamma(w) - \bar{\gamma}}{\sigma^2(w)}. \tag{4}$$

Thus, the within-sample scale-dependence we infer from the tail exponents for the *SOEP* sample is consistent with the notion that efficient financial markets should lead to equilibrating risk-adjusted returns for wealth portfolios. Within-sample scale-dependence can thus be simply explained by heterogeneous risk-preferences. This, however, does no longer hold for between-sample comparisons. Given the fact that Zipf's law seems to hold for the highest wealth regions in the *manager magazin* sample but not within the survey study, the data at face-value seem to imply i) that scale-dependence exists between samples given the difference in tail exponents and ii) that after a certain threshold, this scale-dependence ceases to exist, as Zipf's and consequently Gibrat's law holds for the richest. This impression is confirmed by the differences in expected returns and the measured variance. Both the non-parametric tests as well as the parameter estimates for the Laplacian growth rate distribution indicate that while the expected returns increase between the samples with increasing wealth, the variance for the growth process decreases. It thus seems that the excess returns of the richest individuals in Germany cannot be explained by a higher risk-preference of this subset of the population, as this should be reflected in the variance of returns to net wealth.

The "Data First" interpretation thus suggests not only scale-dependence but scale-dependence that cannot be explained by heterogeneous risk-preferences alone. To explain the estimation results within the framework of a random-growth process, we need to assume that financial markets are not fully competitive in a conventional notion. This would suggest that investors' talent or the increased set of possibilities that comes with being very wealthy enables the richest to persistently beat the market and achieve above-average risk-adjusted returns. Notice that this finding also is at odds with the conventional

[10] There is, however, an ongoing debate on whether this relationship can be established empirically (French et al., 1987; Campbell, 1987; Nelson, 1991; Campbell and Hentschel, 1992; Harvey, 2001; Goyal and Santa-Clara, 2003; Brandt and Kang, 2004; Ghysels et al., 2005; Bali and Peng, 2006; Andersen et al., 2006; Guo and Whitelaw, 2006; Lundblad, 2007; Bali, 2008; Gonzales et al., 2012). While most of the studies find at least weak support for the risk-return trade-off for various time frames and markets, the debate seems to have now shifted to the precise functional form of this relationship - as opposed to the linear one implied by the CAPM.

wisdom on risk-preferences stating that, empirically, the degree of risk-aversion decreases in wealth levels which should lead to higher variances in returns to portfolios with higher net worth (Guiso et al., 1996; King and Leape, 1998; Calvet and Sodini, 2014).

4.3 Theory First

The estimates of the tail exponent for the *manager magazin* and *SOEP* data likely suffer from two different potential sources of bias: The equiprobable sampling of the *SOEP* makes it very unlikely (indeed close to impossible) that the maximum wealth levels are included. To give some intuition on the orders of magnitude involved here, the probability of including a unique maximum wealth level for the *SOEP* sampling ratio under equiprobable sampling would be 0.035 per-cent and thus would be essentially equal to zero. Adding to this problem are concerns of social desirability biases, in particular that the very rich tend to not respond to survey requests. As the probability of non-response is therefore positively correlated to the wealth levels, it is likely that the *SOEP* suffers from *differential non-response* (Kennickell and Woodburn, 1997; Eckerstorfer et al., 2016). These two considerations imply that the *SOEP* data likely suffer from *undersampling*, that is, the maximum wealth levels are not included at all in the sample. In contrast to that, the *manager magazin* sample is a carefully selected sample particularly aimed at describing the richest individuals in Germany. Thus, one could expect that undersampling is not that much of a problem for this sample. However, as the *manager magazin* staff relies on public records for their rich list, they likely underestimate the actual wealth levels for the richest 500 Germans due to privacy considerations and tax avoidance that is particularly pronounced for the wealthiest (Alstadsæter et al., 2019). Thus, the *manager magazin* data are expected to suffer from *underreporting*, not undersampling. In somewhat more colloquial terms, the upwards bias in the *SOEP* data is due to the fact that the richest are not included at all, while the upwards bias in the *manager magazin* data is caused by the fact that the richest are not included with the full extent of their wealth.

However, since the data on undersampling or underreporting rates are impossible to estimate by the very nature of the problem, we consider three stylized scenarios on how strongly these two phenomena bias the estimate for the tail exponent: i) unanimous (proportional) underreporting, ii) undersampling and iii) differential (proportional) underreporting. First, we consider the case of unanimous (proportional) underreporting, that is, all respondents only report a fraction r of their wealth. Call this fraction the *response-rate*. As we show in Appendix H, this leads to an unbiased estimator and thus, underreporting would not be a problem for the estimation of inequality in this stylized scenario. Consider secondly the case that is likely pertaining the estimation for the *SOEP* data - undersampling. In our stylized scenario, this would correspond to a case where the upper p-quantile is non-respondent and the wealth distribution is therefore p-truncated. As we show in Appendix H, this asymptotically leads to a (strong) upwards bias for the MLE of the tail exponent. Finally, we also consider the case of differential (proportional) underreporting. Since unanimous (proportional) underreporting leads to unbiased estimators, the only source of upwards bias would be differential underreporting of the richest in the sample. For this, we consider the case where the upper p-quantile of the wealth distribution only reports a fraction of their wealth with a reporting rate of r. Indeed, for all $p \in (0, 1)$ and $r \in (0, 1)$, this leads asymptotically to upwards biased estimators. Thus, in the case of underreporting, the differential behaviour of the richest compared to the relatively less wealthy is necessary to cause upwards biases in our stylized scenario.

Finally, we consider the relative bias for the two cases of differential undersampling and underreporting. For this, we plot the upwards deviation from the theoretically expected tail exponent of $\alpha = 1$ for $p \in (0, 0.2)$ as a realistic quantile of affected individuals and different reporting rates. The case of $r = 0$ obviously corresponds to the case of undersampling.

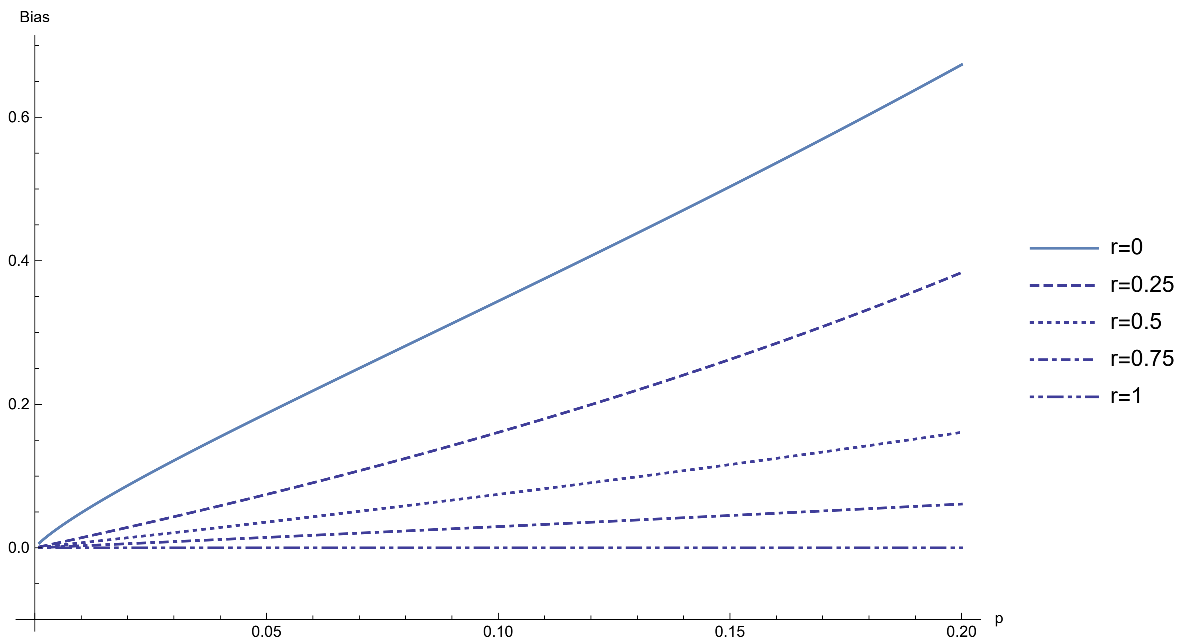

Figure 5. Upwards bias resulting from different combinations of reporting rates r and fractions of affected wealth levels p compared to the theoretically expected α of unity for Zipf's law.

As can be seen in Figure 5, we get the somewhat intuitive result that the relative bias is decreasing in the reporting rate r, since for a smaller r, a larger fraction of wealth is not reported. For $r = 1$, this corresponds to the initial distribution and the bias is thus 0 for all p. However, what might be not that intuitive is the relative strength of the bias by undersampling compared to underreporting. If we allow only a fraction of 25% of wealth to be reported by the richest p-quantile, this leads to a disproportionately much smaller bias on the estimator. This result leads us to conclude that indeed the estimations from the *manager magazin* sample are probably much less (upwards) biased compared to the estimates from the *SOEP* sample.

Differential reporting is, however, also completely consistent with downwards biased estimates. For this, consider the fact that the tendency to round or the "digit-preference" is probably much more pronounced for lower levels of wealth than for the more salient, higher levels. Indeed, it seems quite likely that data collection from the manager magazin staff suffers from a salience bias , that is, much more effort is put into acquiring information on the richest individuals (Balz et al., 2014). Thus, as a more diverse set of sources is considered, the reported wealth levels are probably higher on average for this salient richest set. Since the $r > 0$ parameter only describes the relative reporting rate of the highest p quantile relative to the remaining $1 - p$ share of the population, $r > 1$ is entirely possible. This is the case, whenever the effect of the salience bias outweighs the effects of tax avoidance and other sources of underreporting. Consider the stylized scenario, when the richest 100 individuals are considered to be a salient set which corresponds to $p \approx 0.2$. Assume further that the "true" distribution is governed by Zipf's law. If we estimate the (relative) reporting rates r for the estimated tail exponents and the given $p = 0.2$, the pattern in Table 4 emerges.

Implied Reporting Rates r manager magazin	2010	2011	2013	2014	2015	2016
$\hat{\alpha}$	0.9983	0.9863	1.0999	0.9874	0.9358	0.7615
p	0.2	0.2	0.2	0.2	0.2	0.2
$\hat{r}$	1.0086	1.0719	0.6350	1.0659	1.4092	4.7878

Table 4. Implied reporting rates r for differential underreporting with $p = 0.2$ and Zipf's law.

As can be seen, apart from the 2016 case, the reporting rates seem quite plausible given the intuitive explanation above. Furthermore, the 2016 estimate for the reporting rate seems to truly reflect the fact that there was a qualitative change in the data collection procedure by the *manager magazin* staff. Within this interpretational framework, we can thus speculate that the main cause for the very low α estimate in 2016 is due to the fact that wealth held in foundations is much more salient for the richest part of the population in contrast to other types of wealth. Thus, within this interpretation and apart from the special 2016 case, the assumption of Zipf's and consequently Gibrat's law with scale-independence seems entirely plausible. This impression is of course strengthened by the fact that Zipf's law cannot be rejected in all cases from 2010 to 2015 for the original estimates in Table 2. This implies that the 95 per-cent confidence interval for the reporting rates always spans $r = 1$ also.

The same holds for the *SOEP* sample. Again under the assumption of the true distribution being well described by Zipf's law, we estimate the implied p-quantiles of non-respondents by the estimated tail exponent α as p_{pl}. We then proceed to estimate the size of this quantile of non-respondents relative to the whole sample size by $p_{tot} = p_{pl}/(1 + p_{pl}) \cdot (n_{pl}/n_{tot})$, where n_{pl} is the population within the power law tail and n_{tot} the total sample size. The results are summarized in Table 5.

Implied Non-Response Rates p			
SOEP	2002	2007	2012
$\hat{\alpha}$	1.3144	1.0978	1.2982
n_{pl}	961	1,260	1,332
$\hat{p}_{pl}$	0.0906	0.02310	0.08538
$\hat{p}_{tot}$	0.0033	0.0010	0.0043

Table 5. Implied non-response rates $\hat{p}_{pl}$ and $\hat{p}_{tot}$ for undersampling and Zipf's law.

As the table shows, the implied non-response rates, especially as a ratio of the whole sample, are very small. Arguably, it is very plausible that the effects of equiprobable sampling combined with differential non-response can lead to non-reponse rates p_{tot} of 0.1 to 0.4%. Thus, again, also the survey data seem to be consistent with the interpretation of a scale-independent random growth process and thus, Zipf's law in wealth levels. Since the ensemble of Zipfian samples is also Zipfian, the "Theory First" interpretation thus suggests that the whole multiplicative growth regime in Germany is characterized by scale-independence. The findings of scale-dependence in the "Data First" framework thus are here understood as mere artefacts of differential biases within the differing data collection procedures, such as equiprobable sampling and differential non-response for the survey study and the possible counteracting effects of salience biases, limited information availability and tax avoidance.

While it is, given this interpretation, tiny rates of differential non-response causing the deviation between the estimated tail exponents in the *SOEP*, $\hat{\alpha}^{SOEP}$, and the *manager magazin*, $\hat{\alpha}^{mm}$, the consequences for the estimated total wealth within the power law regime are enormous. We work with the continuous analogue of the power law distribution and simply integrate to derive a measure for the total power law wealth W with

$$\hat{W}_i \quad = \hat{N} \cdot \int_{\hat{w}_{min}^{SOEP}}^{w_{max}^{mm}} \hat{f}_i(w) \cdot w \ dw, \tag{5}$$

where $\hat{f}_i(w)$ denotes the estimated PDF of the power law given by

$$\hat{f}_i(w) \quad = \hat{\alpha}_i \cdot \hat{w}_{min}^{SOEP} \cdot w^{(-\hat{\alpha}_i+1)}, \tag{6}$$

with $\hat{w}_{min}^{SOEP}$ as its respective parameters $\hat{\alpha}_i \in \{\hat{\alpha}^{mm}; \hat{\alpha}^{SOEP}\}$. Notice that the only source of difference here is α_i which can be either the estimate for the survey study or the rich list. $\hat{N}$ is determined according to the procedure in Appendix I and also common between both types of samples. Since the respective periods for which the studies were conducted do never coincide, we consider all possible combinations of parameters for the estimation of the total power law wealth W. The results are given in Table 6.

SOEP $\hat{W}$

mm years / SOEP years	2010	2011	2013	2014	2015	2016
2002	340	314	839	393	279	85
2007	430	395	1,098	495	345	99
2012	303	278	783	349	242	68

mm $\hat{W}$

mm years / SOEP years	2010	2011	2013	2014	2015	2016
2002	1,096	1,079	1,819	1,409	1,280	1,171
2007	1,129	1,109	1,910	1,447	1,305	1,177
2012	1,140	1,118	1,939	1,458	1,313	1,179

Table 6. Estimated total wealth in the power law tail for the *SOEP* and *manager magazin* samples (in billion €, inflation-adjusted with base year 2010 and rounded off to the integer level). The *SOEP estimates* are calculated from the parameter combinations with the $\hat{w}_{min}^{SOEP}$, $\hat{\alpha}^{SOEP}$ and w_{max}^{SOEP} from the *SOEP* sample and the $\hat{N}$ from *manager magazin* sample for the respective years given in the table. The *manager magazin* estimates are calculated from the parameter combinations with the $\hat{w}_{min}^{SOEP}$ from the *SOEP* samples and the $\hat{\alpha}^{mm}$, the in-sample power law population N^{mm} and $\hat{w}_{min}^{mm}$ from the *manager magazin* sample for the respective years given in the table.

As can be easily seen, the difference between the *SOEP* and *manager magazin* estimates is enormous. The total wealth is higher by at least double for the *manager magazin*, with some estimates being more than one order of magnitude larger. While this last finding is restricted to the problematic case of 2016 and might thus be spurious, the ratio for the other years is also huge. For 2015, for example, the *manager magazin* estimates are between 3 to 5.5 times larger than their *SOEP* analogue. Given the fact that these estimates only differ by their respective estimated tail exponents, this highlights the highly non-linear effects of biases in the estimation of this tail exponent on estimated total wealth. We note further that these estimates taken on their own are likely downwards biased even for the *manager magazin* case. This is due to the fact that we take the empirical maximum wealth levels (from the rich list) that might be subject to underreporting and tax avoidance. While the total wealth estimates on their own should thus be taken with a grain of salt, the methodological point on their relative size estimated from different sample types is arguably more relevant. If the "Theory First" interpretation is even partially correct, this implies that wealth estimates from survey studies might be severely distorted and downwards biased both with respect to the within-sample inequality as well as the estimates for total wealth.

5 Discussion

Are returns to net wealth scale-dependent for the richest Germans? While the "Data First" interpretation suggests that the observed consistent deviations from Gibrat's law in the growth process and consequently Zipf's law in wealth levels are reflective of the true data-generating process, the "Theory First" interpretation is able to plausibly explain these deviations by differential biases between sample types. The deliberately unsatisfactory answer we aim to give is that we do not know and cannot decide with the given evidence. Indeed, we consider it to be the main contribution of this paper to point this out.

Three implications follow from the apparent lack of decisive evidence in favour or against scale-dependence. First,starting from the most general one, our result seems to be an instance of the underdetermination of scientific theory by evidence featuring prominently in the philosophy of science at least since the turn of the 20th century (Quine, 1975). As we argue, the proposed mutually contradicting interpretations are observationally equivalent with each other. Appealing to plausibility as a criterion also seems to be no solution, as both the implied parameter values within the "Theory First" framework as well as the narrative underlying the "Data First" interpretation both appear to be perfectly plausible. The pre-analytic vision thus not only informs the research agenda but also the conclusions drawn from the results.

Second, in a more narrow sense, our results can also be interpreted as a cautionary tale against the pitfalls of comparing trends between and pooling data from different sample types potentially affected by differential biases. This is at least what "Theory First" suggests. Given the huge difference in implied total wealth from potentially a tiny degree of undersampling – an estimated 0.1% to 0.4% non-response rate leading up to one order of magnitude difference in estimated total wealth – the choice of sampling might heavily influence the inferences one can draw from it. This small degree of necessary bias seems to be easily explainable by the impact of equiprobable sampling alone within a power law regime of the population. As we have shown analytically, this leads to a inclusion probability of the maximum potentially heavily affecting the estimation of the tail exponent by only about 0.035%. Apart from such issues pertaining the cross-sectional distributions, the proposed differential biases also cast doubt on the validity of conclusions in the time-series dimension within sample-types. This is due to the fact that valid inferences in the presence of such biases require stability of parameters over time. If parameters are allowed to change over time, identified trends might become spurious and rather reflect changes in bias. Since the proposed explanation of biases is partially behavioural, such as building on empirically well established phenomena such as salience, differential tax avoidance and social desirability rather than institutional, there is no reason to expect stability. The estimated parameters within the "Theory First" framework indeed suggest such variable behavioural responses over time.

Finally and maybe most controversially, the findings within both respective interpretational frameworks challenge a particular piece of established wisdom within the economics profession. Within the "Data First" interpretation, the estimations seem to suggest that the richest individuals are able to circumvent the risk-return trade-off by a huge margin and increase their expected returns while simultaneously decrease their risk-exposure as measured by the variance of returns. This is in partial agreement with Fagereng et al. (2018) and Bach et al. (2017) who also find excess risk-adjusted returns for the wealthiest but find margins that are much more narrow and an increase in risk-exposure of the wealthiest in contrast to our measured decrease. Given this pronounced deviation between the richest sub-groups, this finding also calls the efficiency of capital markets into question that primarily govern the growth process there. With respect to the "Theory First" interpretation, the enormous differences between total wealth estimates suggest that inferences from survey studies regarding the cross-sectional distribution of wealth and its time-variation might be heavily distorted. This also suggests that discussions about the notion of "representativeness" in scale-free systems are not discussions about technical subtleties but disagreements in substance. In Appendix G, we conduct a simple analytical thought experiment for an extreme case of unrepresentative oversampling of the richest, that is, "logarithmic sampling" according to orders of magnitude in wealth levels. We show that for Zipf's law, the necessary sampling ratio to surely include a unique maximum observation decays extremely fast by a power function. If the maximum is indeed as important as our analytical results on the distortion of estimators by undersampling indicate, even conventional oversampling techniques are arguably insufficient and should mimic our stylized scenario of logarithmic sampling more to achieve unbiased estimations. Thus, we would suggest that being representative *about wealth* requires being strongly unrepresentative *about individuals.*

References

Abul-Magd, A. Y. (2002). Wealth distribution in an ancient Egyptian society. *Physical Review E*, 66(5).

Alstadsæter, A., Johannesen, N., and Zucman, G. (2019). Tax evasion and inequality. *American Economic Review*, 109(6):2073–2103.

Andersen, T. G., Bollerslev, T., Christoffersen, P. F., and Diebold, F. X. (2006). Volatility and correlation forecasting. *Handbook of economic forecasting*, 1:777–878.

Aoki, S. and Nirei, M. (2016). Pareto distribution of income in neoclassical growth models. *Review of Economic Dynamics*, 2:25–42.

Bach, L., Calvet, L. E., and Sodini, P. (2017). Rich pickings? Risk, return, and skill in the portfolios of the wealthy. *HEC Paris Research Paper*, 1126:16–03.

Bach, S., Beznoska, M., and Steiner, V. (2011). A wealth tax on the rich to bring down public debt? *DIW Berlin Discussion Paper*, 113:1–22.

Bali, T. G. (2008). The intertemporal relation between expected return and risk. *Journal of Financial Economics*, 8:101–131.

Bali, T. G. and Peng, L. (2006). Is there a risk-return trade-off? Evidence from high-frequency data. *Journal of Applied Econometrics*, 21(8):1169–1198.

Balz, J., Sunstein, C., and Thaler, R. (2014). Choice architecture. In Shafir, E., editor, *The behavioral foundations of public policy*, pages 428–439. Princeton University Press, Princeton.

Benhabib, J. and Bisin, A. (2018). Skewed wealth distributions: Theory and empirics. *Journal of Economic Literature*, 56(4):1261–91.

Benhabib, J., Bisin, A., and Zhu, S. (2011). The distribution of wealth and fiscal policy in economies with finitely lived agents. *Econometrica*, 79(1):123–175.

Bergfeld, M.-M. H. and Weber, F. M. (2011). Dynasties of innovation: Highly performing German family firms and the owners' role for innovation. *International Journal of Entrepreneurship and Innovation Management*, 13(1):80–94.

Bottazzi, G. (2004). Subbotools user's manual. *Laboratory of Economics and Management Working Paper Series, Sant'Anna School of Advanced Studies, Pisa (Italy).*

Brandt, M. W. and Kang, Q. (2004). On the relationship between the conditional mean and volatility of stock returns: A latent VAR approach. *Journal of Financial Economics*, 72(2).

Brzezinski, M. (2014). Do wealth distributions follow power laws? Evidence from 'rich lists'. *Physica A: Statistical Mechanics and its Applications*, 40:155–162.

Calvet, L. E. and Sodini, P. (2014). Twin picks: Disentangling the determinants of risk-taking in household portfolios. *The Journal of Finance*, 69(2):867–906.

Campbell, J. Y. (1987). Stock returns and the term structure. *Journal of Financial economics*, 18(2):373–399.

Campbell, J. Y. and Hentschel, L. (1992). No news is good news: An asymmetric model of changing volatility in stock returns. *Journal of Financial Economics*, 31(3):281–318.

Castaldi, C. and Milaković, M. (2007). Turnover activity in wealth portfolios. *Journal of Economic Behavior & Organization*, 6:537–552.

Champernowne, D. G. (1953). A model of income distribution. *The Economic Journal*, 63(250):318–351.

Clauset, A., Shalizi, C. R., and Newman, M. E. J. (2009). Power-law distributions in empirical data. *SIAM Review*, 5:661–703.

Coelho, R., Néda, Z., Ramasco, J. J., and Augusta Santos, M. M. (2005). A family-network model for wealth distribution in societies. *Physica A: Statistical Mechanics and its Applications*, 35:515–528.

Córdoba, J. C. (2008a). A generalized Gibrat's law. *International Economic Review*, 49(4):1463–1468.

Córdoba, J.-C. (2008b). On the distribution of city sizes. *Journal of Urban Economics*, 63(1):177–197.

Cowell, F. A. (2011). Inequality among the wealthy. *Centre for Analysis of Social Exclusion Series*, 15:1–33.

Davies, J. B. and Shorrocks, A. F. (2000). The distribution of wealth. In Atkinson, A. and Bourguignon, F., editors, *Handbook of Income Distribution*, pages 605–675. Elsevier Science B.V., Amsterdam.

De Haan, L. and Resnick, S. (1997). On asymptotic normality of the Hill estimator. *Communications in Statistics. Stochastic Models*, 14(4):849–866.

Deaton, A. (2003). Saving and Liquidity constraints. *Econometrica*, 59(5):1221–1248.

Drăgulescu, A. A. and Yakovenko, V. M. (2001). Exponential and power law probability distributions of wealth and income in the United Kingdom and the United States study. *Physica A: Statistical Mechanics and its Applications*, 29:213–221.

Dynan, K. E., Skinner, J., and Zeldes, S. P. (2004). Do the rich save more? *Journal of Political Economy*, 112(2):397–444.

Eckerstorfer, P., Halak, J., Kapeller, J., Schütz, B., Springholz, F., and Wildauer, R. (2016). Correcting for the missing rich: An application to wealth survey data. *Review of Income and Wealth*, 62(4):605–627.

Fagereng, A., Guiso, L., Malacrino, D., and Pistaferri, L. (2018). The historical evolution of the wealth distribution: A quantitative-theoretic investigation. *CESifo Working Paper Series*, 7107.

Fama, E. F. (1965). The behavior of stock-market prices. *The Journal of Business*, 38(1):34–105.

Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, 25(2):383–417.

Fama, E. F. (1991). Efficient capital markets: II. *The Journal of Finance*, 46(5):1575–1617.

Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 222(594-604):309–368.

French, K. R., Schwert, G. W., and Stambaugh, R. F. (1987). Expected stock returns and volatility. *Journal of Financial Economics*, 19(1):3–29.

Frick, J. R., Grabka, M. M., and Marcus, J. (2007). Editing and multiple imputation of item-non-response in the 2002 wealth module of the German Socio-Economic Panel (SOEP). *DIW Berlin: Data Documentation*, 18.

Gabaix, X. (1999). Zipf's law for cities: an explanation. *The Quarterly Journal of Economics*, 114(3):739–767.

Gabaix, X. (2009). Power laws in economics and finance. *Annual Review of Economics*, pages 255–293.

Gabaix, X., Lasry, J.-M., Lions, P.-L., and Moll, B. (2016). The dynamics of inequality. *Econometrica*, 84(6):2071–2111.

Ghysels, E., Santa-Clara, P., and Valkanov, R. (2005). There is a risk-return trade-off after all. *Journal of Financial Economics*, 76(3):509–548.

Gibrat, R. (1931). *Les inégalités économiques; applications: aux inégalités des richesses, à la concentration des entreprises, aux populations des villes, aux statistiques des familles, etc., d'une loi nouvelle, la loi de l'effet proportionnel.* Librairie du Recueil Sirey.

Glazer, E. (2016). At J.P. Morgan, $9 million in assets isn't rich enough. *Wall Street Journal*, March, 18.

Goldstein, M. L., Morris, S. A., and Yen, G. G. (2004). Problems with fitting to the power-law distribution. *The European Physical Journal B*, 41(2):255–258.

Gonzales, M., Nave, J., and Rubio, G. (2012). The cross section of expected returns with MIDAS betas. *Journal of Financial and Quantitative Analysis*, 47(1):115–135.

Goyal, A. and Santa-Clara, P. (2003). Idiosyncratic risk matters *The Journal of Finance*, 58(3):975–1007.

Guiso, L., Jappelli, T., and Terlizzese, D. (1996). Income risk, borrowing constraints, and portfolio choice. *The American Economic Review*, pages 158–172.

Guo, H. and Whitelaw, R. F. (2006). Uncovering the risk–return relation in the stock market. *The Journal of Finance*, 61(3):1433–1463.

Harvey, C. R. (2001). The specification of conditional expectations. *Journal of Empirical Finance*, 8(5):573–637.

Hegyi, G., Néda, Z., and Santos, M. A. (2007). Wealth distribution and pareto's law in the Hungarian medieval society. *Physica A: Statistical Mechanics and its Applications*, 38:271–277.

Heitjan, D. F. and Rubin, D. B. (1991). Ignorability and coarse data. *The Annals of Statistics*, 19(4):2244–2253.

Hubmer, J., Krusell, P., and Smith Jr, A. A. (2016). The historical evolution of the wealth distribution: A quantitative-theoretic investigation. *NBER Working Paper Series*, 23011.

Jappelli, T. and Pistaferri, L. (2014). Fiscal policy and MPC heterogeneity. *Macroeconomics*, 6(4):107–136.

Jones, C. I. and Kim, J. (2018). A Schumpeterian model of top income inequality. *Journal of Political Economy*, 126(5):1785–1826.

Kennickell, A. and Woodburn, R. L. (1997). Consistent weight design for the 1989, 1992, and 1995 SCFs, and the distribution of wealth. *Federal Reserve Board Survey of Consumer Finances Working Papers*.

Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Mathematica*, 131(1):207–248.

King, M. A. and Leape, J. I. (1998). Wealth and portfolio composition: Theory and evidence. *Journal of Public Economics*, 69(2):155–193.

King, M. R. and Maier, P. (2010). Would greater regulation of hedge funds reduce systemic risk? In Kolb, R. W., editor, *Lessons from the Financial Crisis: Causes, Consequences, and Our Economic Future*, pages 625–632. John Wiley & Sons, Hoboken, New Jersey.

Kotz, S., Kozubowski, T. J., and Podgorski, K. (2001). *The Laplace Distribution and Generalizations*. Birkhauser, Boston.

Levy, M. (2003). Are rich people smarter? *Journal of Economic Theory*, 11:42–64.

Levy, M. and Levy, H. (2003). Investment talent and the Pareto wealth distribution: Theoretical and experimental analysis. *Review of Economics and Statistics*, 85(3):709–725.

Levy, M. and Solomon, S. (1997). New evidence for the power law distribution of wealth. *Physica A: Statistical Mechanics and its Applications*, 24:90–94.

Levy, S. (1998). Wealthy people and fat tails: And explanation for the Lévy distribution of stock returns. *Finance Working Paper: Anderson Graduate School of Management*, 30(98).

Lundblad, C. (2007). The risk return tradeoff in the long run: 1836–2003. *Journal of Financial Economics*, 85(1):123–150.

Luttmer, E. G. (2010). Models of growth and firm heterogeneity. *Annual Review of Economics*, 2(1):547–576.

Luttmer, E. G. J. (2011). On the mechanics of firm growth. *The Review of Economic Studies*, 78(3):1042–1068.

Mandelbrot, B. (1960). The Pareto-Levy law and the distribution of income. *International Economic Review*, 1(2):79–106.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica*, 41(5):867–887.

Milaković, M. (2003). *Towards a Statistical Equilibrium Theory of Wealth Distribution*. PhD thesis, New School for Social Research, 65 Fifth Ave., NY 10003.

Milanovic, B. (2016). *Global Inequality - A New Approach for the Age of Globalization*. Harvard University Press, Cambridge.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica*, pages 347–370.

Ning, D. and You-Gui, W. (2007). Power law tail in the Chinese wealth distribution. *Chinese Physics Letters*, 24(8):2434–2436.

Nirei, M. (2009). Pareto distributions in economic growth models. *Institute of Innovation Research Working Paper*, 09(05).

Piketty, T. and Zucman, G. (2015). Wealth and inheritance in the long run. In *Handbook of Income distribution*, volume 2, pages 1303–1368. Elsevier Science B.V., Amsterdam.

Quadrini, V. (1999). The importance of entrepreneurship for wealth concentration and mobility. *Review of Income and Wealth*, 45(1):1–19.

Quine, W. V. (1975). On empirically equivalent systems of the world. *Erkenntnis*, 9(3):313–328.

Ramanujan, S. (1988). *The lost notebook and other unpublished papers*. Narosa Publishing House, New Delhi, Berlin, New York.

Schneeweiß, H., Komlos, J., and Ahmad, A. S. (2010). Symmetric and asymmetric rounding: a review and some new results. *AStA Advances in Statistical Analysis*, 94(3):247–271.

Simon, H. A. (1955). On a class of skew distribution functions. *Biometrika*, 42(3/4):425–440.

Simon, H. A. and Bonini, C. P. (1958). The size distribution of business firms. *The American Economic Review*, 48(4):607–617.

Sinha, S. (2006). Evidence for power law tail of the wealth distribution in India. *Physica A: Statistical Mechanics and its Applications*, 35:555–562.

Statistisches Bundesamt (2017). Bevölkerung in Deutschland voraussichtlich auf 82,8 Millionen gestiegen. *Pressemitteilung*, 33/17.

Steindl, J. (1965). *Random Processes and the Growth of Firms*. Hafner, New York.

Stumpf, M. P. H. and Porter, M. (2012). Critical truths about power laws. *Science*, 335(6069):665–666.

Subbotin, M. T. (1923). On the law of frequency of error. *Matematicheskii Sbornik*, 31(2):296–301.

Toda, A. A. (2014). Incomplete market dynamics and cross-sectional distributions. *Journal of Economic Theory*, 154:310–348.

Wachter, J. A. and Yogo, M. (2010). Why do household portfolio shares rise in wealth? *The Review of Financial Studies*, 23(11):3929–3965.

Wold, H. O. A. and Whittle, P. (1957). A model explaining the Pareto distribution of wealth. *Econometrica*, 25(4):591–595.

Zipf, G. K. (1949). *Human behavior and the principle of least effort*. Addison-Wesley Press, Boston.

Appendix

A Model

Consider the Markov diffusion with support over the real half-line $(0, \infty)$ for the normalized wealth w of a typical household or individual given by

$$dw_t/w_t = \mu(w)dt + \sigma(W)dB_t, \tag{7}$$

where μ is the mean growth rate of normalized wealth, σ its standard deviation and dB_t are Wiener increments. Denote by $f(w,t)$ the distribution of normalized wealth levels at t and $f(w)$ the stationary density for $t \to \infty$. The Fokker-Planck equation is then given by

$$\frac{\partial}{\partial t} f(w,t) = -\frac{\partial[\mu(w)wf(w,t)]}{\partial w} + \frac{1}{2}\frac{\partial^2[\sigma(w)w^2 f(w,t)]}{\partial w^2}. \tag{8}$$

For the stationary state, it has to hold that

$$0 = -\frac{\partial[\mu(w)wf(w)]}{\partial w} + \frac{1}{2}\frac{\partial^2[\sigma(w)w^2 f(w)]}{\partial w^2}. \tag{9}$$

Integrating yields

$$0 = -[\mu(w)wf(w,t)] + \frac{1}{2}\frac{\partial[\sigma(w)w^2 f(w,t)]}{\partial w}. \tag{10}$$

This allows to solve for $f(w)$ by differentiating the right term by

$$\begin{aligned} 0 &= -\mu(w) \cdot w \cdot f(w) \\ &+ \frac{1}{2}\left[\frac{\partial \sigma^2(w)}{\partial w} \cdot w^2 \cdot f(w) + \sigma^2(w) \cdot 2w \cdot f(w) + \sigma^2 \cdot w^2 \cdot f'(w)\right] \end{aligned} \tag{11}$$

and therefore

$$f(w) = \frac{\sigma^2(w) \cdot w^2 \cdot f'(w)}{2\mu(w)w - (\partial\sigma^2(w)/\partial w)w^2 - \sigma^2(w)w} \tag{12}$$

Assume that the stationary distribution is indeed a power law, that is, assume a reflecting boundary w_{min} which is by Theorem 1 in Levy and Levy (2003) equivalent in this setting. The tail exponent of the stationary density α is then given by

$$\alpha(w,f) = \frac{-w \cdot f'(w)}{f(w)}. \tag{13}$$

Plugging equation (12) in (13) gives the desired result with

$$\alpha(\mu,\sigma) = 1 - 2\frac{\mu(w)}{\sigma^2(w)} + \frac{w}{\sigma^2(w)}\frac{\partial\sigma^2(w)}{\partial w}. \tag{14}$$

Since we consider normalized wealth levels, $\mu(w)$ is given by the excess expected growth rate relative to the average growth rate of all wealth levels $\bar{\gamma}$ by $\gamma(w) - \bar{\gamma}$ which implies for the tail exponent

$$\alpha(\mu, \sigma) = 1 - 2\frac{\gamma(w) - \bar{\gamma}}{\sigma^2(w)} + \frac{w}{\sigma^2(w)}\frac{\partial \sigma^2(w)}{\partial w}.^{\dagger} \tag{15}$$

Zipf's law emerges as a special case of growth rates characterized by Gibrat's law. This implies that the partial $\partial\sigma^2(w)/\partial w$ is zero, as there is no scale-dependence in the variance. Also, Gibrat's law implies that the expected normalized growth rate, that is, the excess growth rate of wealth levels w in relation to the average growth rate, is independent of w for any w and thus, must be zero. This implies indeed the Zipf exponent of $\alpha(0, \sigma) = 1$. To confirm this, consider the general diffusion in equation (7), with $\mu(w) = \mu = 0$ and $\sigma(w) = \sigma$. The general Fokker-Planck equation (8) is for these assumptions given by

$$\begin{aligned} 0 &= -\frac{\partial[0 \cdot w \cdot f(w)]}{\partial w} + \frac{1}{2}\frac{\partial^2[\sigma w^2 f(w)]}{\partial w^2} \\ &= \frac{1}{2}\frac{\partial^2[\sigma w^2 f(w)]}{\partial w^2}. \end{aligned} \tag{16}$$

It is easy to see that a density $f(w)$ solves equation (16), whenever the differentiated term in (16) is independent of w. This is exactly the case for $f(w) = C/w^2$, that is, Zipf's law with a normalizing constant C independent of w, where equation (16) becomes

$$0 = \frac{1}{2}\frac{\partial^2[\sigma w^2 C w^{-2}]}{\partial w^2} \tag{17}$$

$$= \frac{1}{2}\frac{\partial^2[\sigma C]}{\partial w^2} \tag{18}$$

$$= 0. \tag{19}$$

[†] In Gabaix (1999), there is a minor typographical error on page 757, where the correct expression for the tail exponent in equation (13) should say $\gamma(S)$, not $\zeta(S)$, like for our analogous expression in equation (15).

B Literature Review on Highest Wealth Regions

Author(s)	**Countries**	**Data source**	α **estimate**
Abul-Magd (2002)	Egypt	Data for Ancient Egypt (14th century BC) with the area of houses as proxy for wealth (distribution from excavations)	3.76
Bach et al. (2011)	Germany	Rich list provided by *manager magazin* (300 individuals, 2007)	1.34
Brzezinski (2014)	World, US, Russia, China	Rich lists provided by *Forbes* (World Billionaires for 1996 - 2012, Richest American List for 1988 - 2012, Richest Chinese list 2006 - 2012) and the Russian magazine *Finans* (2004 - 2011)	1.2 and 2 (World), 1.4 to 1.7 (US), 1.6 to 2 (China) and 0.7 to 0.8 (Russia)
Castaldi and Milaković (2007)	US, UK	Rich lists provided by *Forbes* (400 individuals, 1996 - 2004) and *Sunday Times* (1000 individuals, 2001 - 2004)	1.25 to 1.57 (Forbes) and 1.03 to 1.19 (Sunday Times)
Coelho et al. (2005)	UK	Data by the Internal Revenue Service for 2001	1.78
Drăgulescu and Yakovenko (2001)	UK	Data by the Internal Revenue Service for 1996	1.9
Eckerstorfer et al. (2016)	Austria	Data from the *Household Finance and Consumption Survey* of 2011 (2,380 observations)	1.14 to 1.36

Hegyi et al. (2007)	Hungary	Data for the owned land for aristocratic families (1283 observations) in Hungary in the year 1550 (proxy for wealth is the number of owned serf families)	0.92
Levy (2003)	US	Rich list provied by *Forbes* (400 individuals, 1996)	1.35
Levy (1998)	US, UK, France	*Forbes* (400 individuals, 1997), *Sunday Times* (1000 individuals, 1997), Almanac *Quid* (162,370 individuals in the highest wealth region for France)	1.35 (US), 1.06 (UK) and 1.82 (France)
Levy and Solomon (1997)	UK	Data by the Internal Revenue Service for 1970	1.4
Milaković (2003)	Sweden, Belgium, Canada, Denmark, Germany, US, UK, France	Lorenz data, various sources	1.07 to 1.68
Ning and You-Gui (2007)	China	Rich list by the Chinese magazine *New Fortune* for the years 2002 - 2004 (400 observations)	2.285 (2002), 2.043 (2003) and 1.758 (2004)
Sinha (2006)	India	Rich list by the Indian magazine *Business Standard* for 2002 and 2003 (125 observations) and by *Forbes* for 2004 (40 observations)	0.81 (2002), 0.82 (2003) and 0.92 (2004)

Table 7. Literature review on the distributional regularities in the highest wealth regions.

C CCDFs for the manager magazin samples

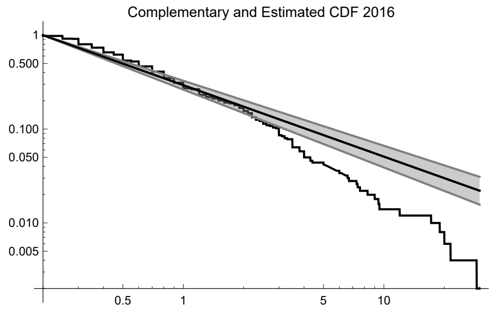

Figure 6. CCDF 2016 (mm sample).

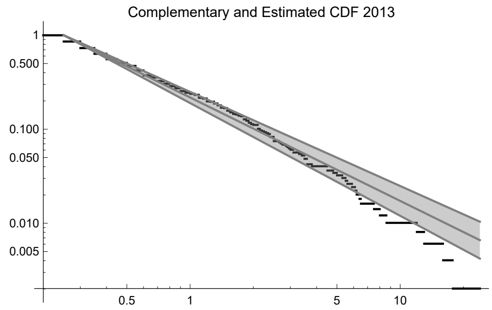

Figure 9. CCDF 2013 (mm sample).

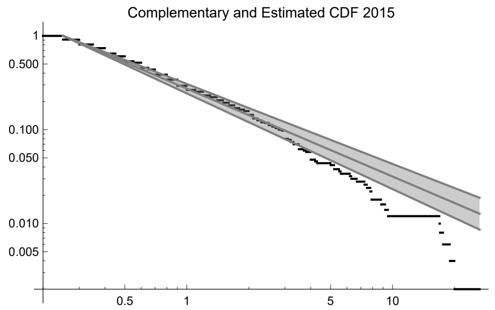

Figure 7. CCDF 2015 (mm sample).

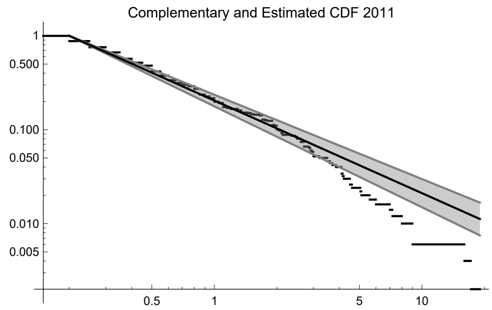

Figure 10. CCDF 2011 (mm sample).

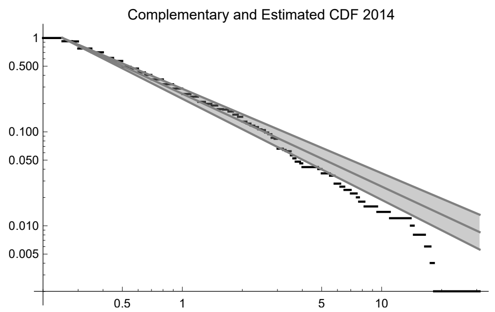

Figure 8. CCDF 2014 (mm sample).

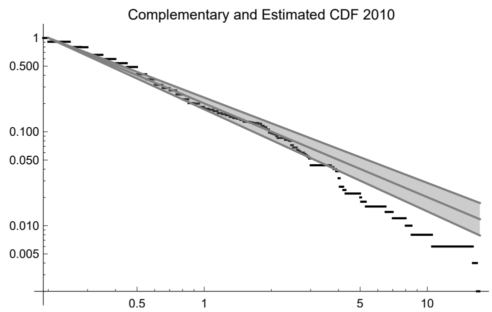

Figure 11. CCDF 2010 (mm sample).

Note: CCDFs on a double-log scale, fits by MLE. Error bands correspond to a deviation of two standard errors for the characteristic exponents. Estimation of the standard errors by approximation from the Gaussianity of the Hill estimator (De Haan and Resnick, 1997).

D CCDFs for the SOEP samples

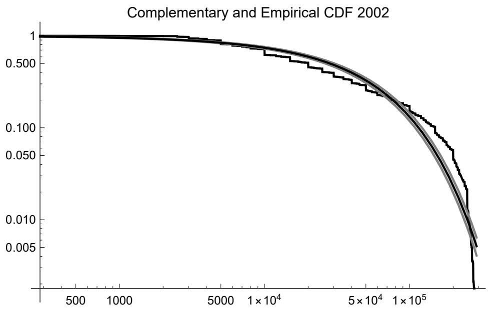

Figure 12. CCDF 2002 (SOEP)
for the lower tail of the distribution.

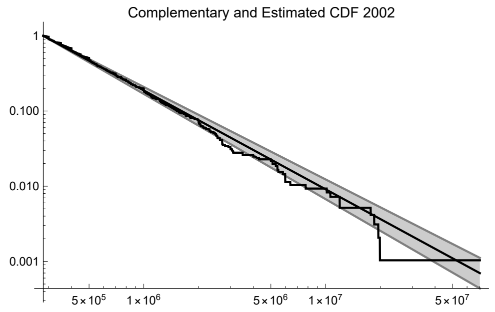

Figure 15. CCDF 2002 (SOEP)
for the upper tail of the distribution.

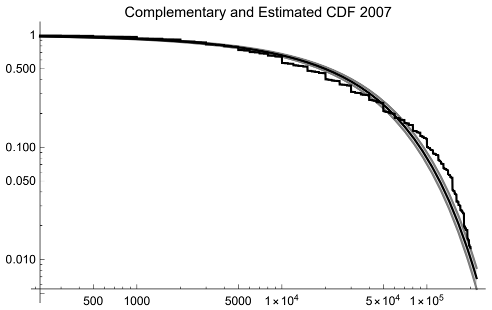

Figure 13. CCDF 2007 (SOEP)
for the lower tail of the distribution.

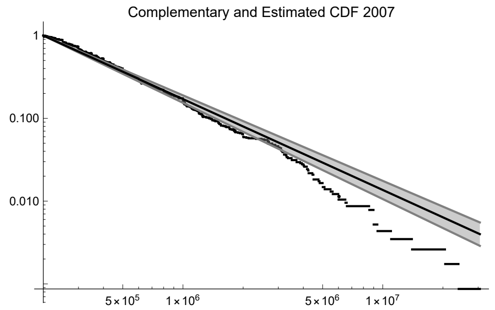

Figure 16. CCDF 2007 (SOEP)
for the upper tail of the distribution.

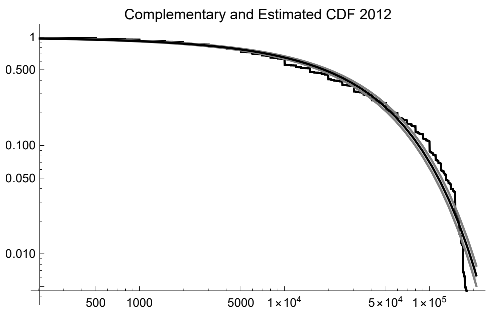

Figure 14. CCDF 2012 (SOEP)
for the lower tail of the distribution.

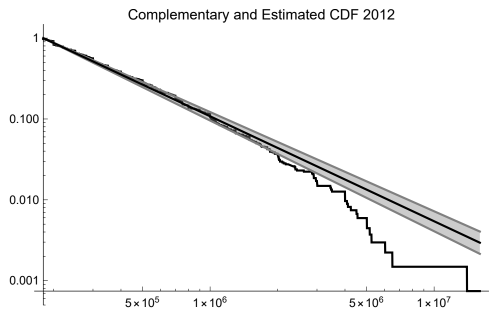

Figure 17. CCDF 2012 (SOEP)
for the upper tail of the distribution.

Note: CCDFs on a double-logarithmic scale, fits by MLE for a Gamma distribution (lower tail) and a power law (upper tail). Error bands correspond to a deviation of two standard errors for the characteristic exponents (in the power law case) and for both parameters simultaneously (in the Gamma case). Estimation of the standard errors in the former case by approximation from the Gaussianity of the Hill estimator (De Haan and Resnick, 1997), in the latter case by utilizing the Fisher information (Fisher, 1922).

E Equivalence Tests for Growth Rate Distributions

Distributional Equivalence Growth Rates (KW)

KW Test	mm 2010/15	mm 2011/16	SOEP 2002/07	SOEP 2007/12
mm 2010/2015	-	0.37	45.8 ***	26.9 ***
	-	(0.543)	(0)	(0)
mm 2011/16	0.37	-	41.4 ***	22.7 ***
	(0.543)	-	(0)	(0)
SOEP 2002/07	45.8 ***	41.4 ***	-	2.16
	(0)	(0)	-	(0.133)
SOEP 2007/12	26.9 ***	22.7 ***	2.16	-
	(0)	(0)	(0.133)	-

Table 8. Test statistics and p-values for the Kruskall-Wallis test of location equivalence. Null hypothesis is location equivalence. * significance at the 10 per-cent, ** at the 5 per-cent and *** at the 1 per-cent level. p-values in parentheses.

Distributional Equivalence Growth Rates (CvM)

CvM test	mm 2010/15	mm 2011/16	SOEP 2002/07	SOEP 2007/12
mm 2010/2015	-	0.152	5.15***	2.98***
	-	(0.385)	(0)	(0)
mm 2011/16	0.152	-	4.73***	2.62***
	(0.385)	-	(0)	(0)
SOEP 2002/07	5.15***	4.73***	-	0.363*
	(0)	(0)	-	(0.0908)
SOEP 2007/12	2.98***	2.62***	0.363*	-
	(0)	(0)	(0.0908)	-

Table 9. Test statistics and p-values for the Cramér-von-Mises test of distributional equivalence. Null hypothesis is distributional equivalence. * significance at the 10 per cent, ** at the 5 per cent and *** at the 1 per cent level. p-values in parentheses.

Distributional Equivalence Growth Rates (KS)

KS test	mm 2010/15	mm 2011/16	SOEP 2002/07	SOEP 2007/12
mm 2010/2015	-	0.0723	0.347***	0.274***
	-	(0.376)	(0)	(0)
mm 2011/16	0.0723	-	0.352***	0.28***
	(0.376)	-	(0)	(0)
SOEP 2002/07	0.347***	0.352***	-	0.145*
	(0)	(0)	-	(0.0604)
SOEP 2007/12	0.274***	0.28***	0.145*	-
	(0)	(0)	(0.0604)	-

Table 10. Test statistics and p-values for the Kolmogorov-Smirnov test of distributional equivalence. Null hypothesis is distributional equivalence. * significance at the 10 per cent, ** at the 5 per cent and *** at the 1 per cent level. p-values in parentheses.

F Subbotin Estimates for the Growth Rate Distributions

Estimated Parameter	$\hat{\kappa}$	SE
SOEP		
2002/07	1.18	0.1766
2007/12	0.8463	0.1178
manager magazin		
2010/15	0.8523	0.08587
2011/16	0.9509	0.09798

Table 11. Summary of estimates for the Subbotin shape parameter and all growth rate distributions. Estimates by MLE, standard errors from the Fisher information (Fisher, 1922). The estimation was done via *Subbotools* 1.3.0 that delivered the most accurate and efficient estimates in simulation runs (Bottazzi, 2004).

G Sampling From a Power Law

Simple Random Sampling without Replacement. Equiprobable Selection of Elements.

Let $N \in \mathbb{N}^+$ denote the total population and $n \in \mathbb{N}^+$, with $N \geqslant n$, the size of the sample. The sampling procedure selects each element of the set M with equal probability and without replacement. If the maximum value of N, denoted by w_{max}, is unique, the probability of w_{max} to be included in the sample, that is, $p(w_{max} \in m)$, is equivalent to the probability of any unique element to be chosen under these conditions. The inclusion probability of w_{max} is therefore given by

$$p(w_{max} \in m) = \frac{\binom{N-1}{n-1}}{\binom{N}{n}} \tag{20}$$

$$= \frac{\dfrac{(N-1)!}{(n-1)! \cdot (N-n)!}}{\dfrac{N!}{n! \cdot (N-n)!}}$$

$$= \frac{n! \cdot (N-1)!}{N! \cdot (n-1)!}$$

$$= \frac{n! \cdot (N-1)!}{N! \cdot (n-1)!}$$

$$= \frac{n \cdot (n-1)! \cdot (N-1)!}{N \cdot (n-1)! \cdot (N-1)!}$$

$$= \frac{n}{N}. \tag{21}$$

The inclusion probability under simple random sampling without replacement for w_{max} therefore corresponds to the sampling ratio $\frac{n}{N}$ and is equal to unity only if $n = N$.

Logarithmic Random Sampling without Replacement. Assumption of Zipf's Law.

Let again $N \in \mathbb{N}^+$ denote the total population and $n \in \mathbb{N}^+$, with $N \geqslant n$, the size of the sample. Furthermore, assume that the total population is now divided into s different "slices", where the length of each slice corresponds to one order of magnitude of the relevant quantity w (that is, the slices are scaled logarithmically). Now the same procedure as above is applied to each logarithmic slice - every element in each slice is selected with equal probability and without replacement. It has to hold that $s \in \mathbb{N}^+$.

For every slice, $\frac{n}{s}$ elements are included in the sample of size n, where n obviously needs to be an integer multiple of s, since $\frac{n}{s} \in \mathbb{N}^+$. The slice covering the highest order of magnitude for w also has to include w_{max} as the maximum value. If one assumes Zipf's law to hold, this range of w includes a proportion 10^{-s+1} of the total population N. Therefore, with this procedure and under the assumption of Zipf's law to hold, $\frac{n}{s}$ elements out of a set of size $\frac{N}{10^{s-1}}$ are chosen. The probability of a unique w_{max} to be included in the chosen m is therefore given by

$$p(w_{max} \in m) = \frac{\binom{(N/10^{s-1})-1}{(n/s)-1}}{\binom{N/10^{s-1}}{n/s}} \tag{22}$$

$$= \frac{\dfrac{((N/10^{s-1}) - 1)!}{((n/s) - 1)! \cdot ((N/10^{s-1}) - (n/s))!}}{\dfrac{N/10^{s-1}!}{(n/s)! \cdot ((N/10^{s-1}) - (n/s))!}}$$

$$= \frac{(n/s)! \cdot ((N/10^{s-1}) - 1)!}{(N/10^{s-1})! \cdot ((n/s) - 1)!}$$

$$= \frac{(n/s) \cdot ((n/s) - 1)! \cdot ((N/10^{s-1}) - 1)!}{(N/10^{s-1}) \cdot ((N/10^{s-1}) - 1)! \cdot ((n/s) - 1)!}$$

$$= \frac{10^{s-1}}{s} \cdot \frac{n}{N}, \qquad \text{with } \frac{n}{N} \geqslant \frac{10^s}{s}.$$

Therefore, the inclusion probability $p(w_{max} \in n)$ under logarithmic sampling compared to simple random sampling converges much faster to unity with the sampling ratio, that is, by a factor of $\frac{10^{s-1}}{s}$. The sampling ratio $\frac{n}{N}$ has to equal merely $\frac{s}{10^{s-1}}$ for $p(w_{max} \in n) = 1$. For $s = 2$, it has to equal $\frac{1}{5}$, for $s = 3$, it has to equal $\frac{3}{100}$, and so on. Of course, this is the case, because for this condition to hold, every element in the "slice" covering the highest order of magnitude for w has to be included in the sample. For $s = 1$, this procedure obviously corresponds to the case for equiprobable sampling, where the inclusion probability equals the sampling ratio $\frac{n}{N}$ (as $\frac{10^{s-1}}{s} = 1$ for $s = 1$).

H Analytical Results for the Estimation of α under p-Truncation

Zipf's Law and Hill Estimator. Preliminaries.

Suppose, a certain discrete quantity w is perfectly distributed according to Zipf's Law. This implies that the α of the underlying power law distribution has to equal unity. According to the *Rank-Size* formulation, its values are therefore given by

$$w(s) = \frac{w_{max}}{s}, \tag{23}$$

with $k = 1, 2, ..., N$ as the respective ranks of a given w in a descending order, N as the number of values with $N \in \mathbb{N}^+$ and w_{max} as the maximum value in the distribution. Equivalently, rewriting equation 23 in terms of the minimum value w_{min} yields

$$w(s) = \frac{w_{min} \cdot N}{s}, \tag{24}$$

since it has to hold that $w_{max} = N \cdot w_{min}$.

A *Maximum Likelihood Estimation* (MLE) for any given (continuous) power law yields the *Hill estimator* (Clauset et al., 2009), that is,

$$\hat{\alpha}(w_{min}; N) = N \cdot \left(\sum_{s=1}^{N} ln\left(\frac{w(s)}{w_{min}} \right) \right)^{-1} \tag{25}$$

which is by equation (24) given by

$$= N \cdot \sum_{s=1}^{N} ln\left(\frac{N}{s} \right) \Bigg)^{-1}, \tag{26}$$

which is now independent of w_{min} and converges asymptotically for $N \to \infty$ to $\alpha = 1$ as the tail exponent for Zipf's law.

Unanimous Proportional Underreporting. Unbiasedness Result.
Suppose that a discrete quantity is perfectly distributed according to Zipf's law. All individuals report only a fraction r of this quantity (the response-rate) which implies unanimous (proportional) underreporting. The rank-size rule for unanimous underreporting thus reads

$$w^{uu}(s) = r \cdot \frac{w_{min} \cdot N}{s}, \tag{27}$$

with $s = 1, 2, ..., N$ as the ranks.
Notice that we also require $w^{uu}_{min} = r \cdot w_{min}$ by $w(N) = w_{min}$ and $w^{uu}(N) = w^{uu}_{min}$.
The Hill-estimator for α^{uu} under unanimous underreporting by equation (27) thus reads

$$\hat{\alpha}^{uu}(w^{uu}_{min}; N) = N \cdot \left(\sum_{s=1}^{N} ln\left(\frac{w^{uu}(s)}{w^{uu}_{min}} \right) \right)^{-1} \tag{28}$$

$$= N \cdot \left(\sum_{s=1}^{N} ln\left(\frac{r \cdot w_{min} \cdot N}{r \cdot w_{min} \cdot s} \right) \right)^{-1} \tag{29}$$

$$= N \cdot \sum_{s=1}^{N} ln\left(\frac{N}{s} \right) \Bigg)^{-1} \tag{30}$$

which is the unbiased estimator of equation (25).

Differential Non-Response of the Upper p Quantile. Asymptotic Properties.
Suppose that a discrete quantity w is perfectly distributed according to Zipf's Law and is p-truncated. The *p-truncated* rank-size rule therefore reads

$$w(s) = \frac{w_{min} \cdot N}{s}, \tag{31}$$

now with $s = \lfloor p \cdot N \rceil + 1, \lfloor p \cdot N \rceil + 2, ..., N$ as the ranks.

For the p-truncated distribution, the MLE for $\hat{\alpha}^{nr}$ as the estimated value resulting from differential non-response would therefore read

$$\hat{\alpha}^{nr}(p; N) = (N - \lfloor p \cdot N \rceil + 1) \cdot \left(\sum_{s=\lfloor p \cdot N \rceil + 1}^{N} ln\left(\frac{N}{s} \right) \right)^{-1}. \tag{32}$$

Further simplifying equation 32 yields

$$\hat{\alpha}^{nr}(p; N) = \frac{1 + N - N \cdot p}{ln \ \frac{N^{N - N \cdot p} \cdot (Np)!}{N!}}. \tag{33}$$

Utilizing Stirling's approximation, in particular Ramanujan's formula which states that $ln \ n! \approx n \cdot ln \ n - n + \frac{1}{6} ln(n(1 + 4n(1 + 2n))) + \frac{1}{2}\pi$ (Ramanujan, 1988), equation 33 now becomes

$$\hat{\alpha}^{nr}(p; N) \approx \frac{1 + N - N \cdot p}{r}. \tag{34}$$

with $r = (N - N \cdot p) \cdot ln \ (N) - \left[N \cdot ln \ N - N + \frac{1}{6} ln(N(1 + 4N(1 + 2N))) \right] + \left[N \cdot p \cdot ln \ (N \cdot p) - N \cdot p + \frac{1}{6} \ ln(N \cdot p(1 + 4(N \cdot p)1 + 2 \cdot (N \cdot p)))) \right]$.¶¶

¶¶In particular, Ramanujan shows that the asymptotic error for the above approximation is $\frac{1}{1440 \ N^3}$ which should be sufficient for the purposes of this formula.

Finally, taking the limit from the expression in equation 34 yields

$$\lim_{N\to\infty} \frac{1+N-N\cdot p}{s} = \frac{1-p}{1-p+p\cdot ln\ \ (p)}. \tag{35}$$

In the limit, the impact of N has completely vanished and the distortion of α is now only dependent on p. As one can easily see, even for large values of N, the estimator is (upwards) biased for any positive p, since for any $p > 0$, the numerator is larger than the denominator which implies an $\alpha > 1$. The result in equation H shows that the upwards bias is not a mere artefact of sample size, but hold true for any sufficiently large N.

Differential Underreporting of the Upper p Quantile. Asymptotic Properties.
Suppose that a quantity w is distributed according to Zipf's law. Consider the case, where only the upper p-quantile is proportionally underreporting by a reporting rate $r \in (0,1)$. The rank-size rule is now a piecewise function for the upper p-quantile and the rest and given by

$$w(s)^{du} = r \cdot \frac{w_{min}\cdot N}{s}, \tag{36}$$

with $s = 1, 2, ..., \lfloor p\cdot N \rceil$ as the ranks and

$$w(s)^{du} = \frac{w_{min}\cdot N}{s} \tag{37}$$

with $s = \lfloor p\cdot N \rceil + 1, \lfloor p\cdot N \rceil + 2, ..., N$ as the remaining ranks.

We require that w_{min}, the minimum of the unchanged initial distribution stays the minimum for the distribution with differential underreporting to avoid issues with the ML estimator which is based on this minimum. For this, the smallest reported value in the underreporting region has to be greater than w_{min}, that is,

$$w(pN)^{du} = r\cdot\frac{w_{min}\cdot N}{pN} > w_{min}$$

and therefore

$$\frac{r}{p} > 1. \tag{*}$$

Thus, for the minimum not to be affected, it has to hold by (*) that the reporting rate exceeds the affected population share of the highest wealth levels. By the linearity of the sum function and assuming condition (*) to hold, we can write the Hill estimator for the tail exponent $\hat{\alpha}^{du}$ by differential (proportional) underreporting now as

$$\hat{\alpha}^{du}(p;r;N) = N\Bigg(\sum_{s=\lfloor p\cdot N\rceil+1}^{N} ln\left(\frac{N}{s}\right)^{-1} + \sum_{s=1}^{\lfloor p\cdot N\rceil} ln\left(\frac{r\cdot N}{s}\right)^{-1}\Bigg). \tag{38}$$

Further simplifying yields

$$\hat{\alpha}^{du}(p;r;N) = \frac{N}{ln\left(\frac{N^{N-Nq}(Nq)!}{N!}\right) + ln\left(\frac{(Nr)^{Nq}}{(Nq)!}\right)}. \tag{39}$$

Utilizing again Stirling's approximation, we get

$$\hat{\alpha}^{du}(p;r;N) \approx \frac{N}{v}, \tag{40}$$

with $v = -\frac{1}{6} ln\left(8N^3 + 4N^2 + N + \frac{1}{30}\right) + N\ p\ ln(Nr) + (N - Np)\ ln(N) + N + N(-ln(N)) - \frac{ln(\pi)}{2}$. Taking the limit for $N \to \infty$ gives

$$\lim_{N\to\infty} \frac{N}{v} = \frac{1}{1 + p\ ln(r)}. \tag{41}$$

Notice that for $p \in (0, 1)$ and $r \in (0, 1)$, the estimator is therefore always upwards biased compared to the benchmark of $\alpha = 1$. Condition (*) precludes the possibility of a negative induced bias which would result from $p\ ln(r) > -1$ and would be uninterpretable. For this, note that condition (*) implies $\ln(r) > \ln(p)$, since $\ln(\cdot)$ is monotonically increasing in its argument. It is thus sufficient to show that $p\ ln(p) > -1$. Rearranging yields

$$\ln(p) + \frac{1}{p} > 0. \tag{42}$$

Define $f(p) = \ln(p) + \frac{1}{p}$. By

$$f(1) = 1 \tag{43}$$

$$\text{and} \quad \frac{df}{dp} = \frac{1}{p} - \frac{1}{p^2} < 0\ , \forall p \in (0, 1), \tag{44}$$

we know that the function is monotonically decreasing for the whole considered interval and positive at the upper interval boundary. From (43) and (44), we can therefore conclude that $f(p) > 0$ for all $p \in (0, 1)$. This is exactly the non-negativity constraint in (42) and thus the desired result that condition (*) implies strictly non-negative induced biases in approximation (41).

I Population Estimates

We estimate the different population levels by taking the CDF $P(w; w_{min}, \alpha)$ of a continuous power law with parameters w_{min}^{SOEP} from the *SOEP* samples and the α^{mm}, the in-sample power law population N^{mm} and w_{min}^{mm} from the *manager magazin* sample. $\hat{N}$ for a specific parameter combination is then calculated as $\hat{N} = 1/(1 - P(w_{min}^{mm}; w_{min}^{SOEP}, \alpha^{mm}) \cdot N^{mm}$. To give some intuition for this, $\hat{N}$ would thus correspond to the power law population, when the power law in the *manager magazin* sample would be extended to the minimum determined in the *SOEP*. The results are given in the table below.

Population $\hat{N}$						
	2010	2011	2013	2014	2015	2016
$\hat{N}$ (w_{min} = 280,000)	352,534	324,983	869,368	407,397	288,683	88,307
$\hat{N}$ (w_{min} = 200,000)	493,270	452,876	1,258,710	567,946	395,524	114,097
$\hat{N}$ (w_{min} = 180,000)	547,981	502,467	1,413,360	630,216	436,510	123,628

Table 12. The estimates are calculated from the parameter combinations with the w_{min}^{SOEP} from the *SOEP* samples and the α^{mm}, the in-sample power law population N^{mm} and w_{min}^{mm} from the *manager magazin* sample for the respective years given in the table.