

Stability and Cycles in a Neo-Kaleckian Model with Autonomous Demand

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Abstract

This paper investigates the dynamic properties of a neo-Kaleckian model with autonomous demand. Building on the frameworks of Allain (2015) and Lavoie (2016), it departs from 2D formulations by allowing capacity utilisation to adjust through a Keynesian quantity-adjustment mechanism, whereby excess demand in the goods market is absorbed via changes in output rather than prices. The resulting three-dimensional system features a locally asymptotically stable interior equilibrium. As the responsiveness of investment to capacity utilisation reaches a critical threshold, the equilibrium loses stability via a Hopf bifurcation, leading to persistent endogenous oscillations in capacity utilisation, growth expectations, and autonomous demand. These results speak directly to the critique advanced by Skott (2017), which emphasises the restrictive conditions required for asymptotic stability in a 2D autonomous-demand-led growth model. By contrast, we show that the interaction of Harroddian forces with a quantity-adjustment mechanism need not produce explosive instability, but may instead sustain stable endogenous cycles around a well-defined long-run growth path.

Keywords: Neo-Kaleckian model, Sraffian supermultiplier, Autonomous demand, Harroddian instability, Hopf bifurcation, Endogenous cycles.

1 Introduction

The problem of growth and cycles has been a central and persistently vexing issue in macroeconomic theory since its inception. The seminal contributions of Harrod (1939) and Domar (1946) revealed a fundamental fragility of capitalist economies by identifying two distinct yet closely related difficulties. The first concerns the absence of any automatic tendency for the warranted and natural rates of growth to coincide. Since parameters determine both, there is no reason to expect them to coincide except by accident, rendering the equilibrium growth path inherently knife-edged. The second, and more troubling, is Harroddian instability: any divergence of the actual growth rate from

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its warranted counterpart sets in motion cumulative forces that drive the economy further away from equilibrium, giving rise to explosive expansions or persistent stagnation.

Earlier Cambridge authors sought to reconcile growth and accumulation through endogenous changes in income distribution.¹ By contrast, in the neo-Kaleckian framework, discrepancies between saving and investment are accommodated through changes in capacity utilisation rather than through adjustments in income distribution.² However, the very mechanism that guarantees stability, namely endogenous capacity utilisation, undermines the model's long-run coherence.

In these models, the equilibrium rate of utilisation may persistently diverge from firms' normal or desired rate, implying that the economy operates indefinitely away from its planned degree of capacity utilisation. In an attempt to address this issue, Amadeo (1986) assumed an investment function in which accumulation reacts positively to deviations of actual utilisation from its normal level, partially introducing an accelerator mechanism into the standard neo-Kaleckian framework³.

Nevertheless, the possibility of persistent deviations of actual from normal utilisation remained a central source of controversy. This feature has been the subject of sustained criticism within the heterodox literature, beginning with Comitteri (1986) and reiterated more recently by Skott (2010, 2012, 2016), Duménil and Lévy (2014), and Cesaratto (2015). Despite important differences in analytical perspective, these authors share the view that a coherent long-run equilibrium requires the actual rate of capacity utilisation to gravitate toward, or converge to, its normal level⁴.

A potential resolution to this impasse emerged from a parallel line of research: the Sraffian Supermultiplier model (SSM) developed by Serrano (1995). Embedded in the Classical-Keynesian tradition, the SSM offers a fully demand-led growth framework in which a component of autonomous demand—growing independently of capital accumulation—acts as the ultimate driver of long-run growth. By construction, this mechanism guides capacity utilisation toward its normal level. More recently, Kaleckian authors such as Allain (2015) and Lavoie (2016) explicitly incorporated the Sraffian supermultiplier mechanism into the neo-Kaleckian framework.

By allowing firms' long-run growth expectations to adjust gradually to the growth rate of autonomous demand, the neo-Kaleckian model with autonomous demand offers

¹The Kaldorian and Robinsonian contributions are often grouped within the broader Cambridge tradition and are sometimes regarded as precursors to later Kaleckian models. In the Kaldor–Robinson framework, however, adjustment operates primarily through changes in income distribution associated with variations in the profit rate.

²Our interpretation follows Hein (2017, p. 230), who notes that, in both the neo-Kaleckian model and the post-Kaleckian variant developed by Bhaduri and Marglin (1990), the rate of capacity utilisation becomes the endogenous variable that reconciles saving and investment. A similar interpretation is provided by Serrano and Freitas (2017), who characterise the Kaleckian closure as relying on the endogenous adjustment of capacity utilisation.

³Despite incorporating an accelerator-type effect through the responsiveness of investment to capacity utilisation, Amadeo (1986) circumvents Harroddian instability insofar as firms do not systematically revise their expected secular growth rate in response to persistent deviations of utilisation from its normal level.

⁴Some alternative attempts were made to address the lack of convergence of capacity utilisation in Kaleckian growth models. One line of argument, associated with Dutt (1997) and Hein (2014), maintains that the Kaleckian long run should be interpreted as a sequence of short-run equilibria, thereby dispensing with the requirement that the actual rate of utilisation converge to a normal level. A different strategy, proposed by Nikiforos (2018), consists of endogenising the normal rate of utilisation itself, allowing it to adjust to realised utilisation over time. This approach, however, has been critically assessed by Girardi and Pariboni (2019), who show that treating normal utilisation as the adjusting variable gives rise to microeconomic inconsistencies, particularly regarding firms' pricing and accumulation decisions.

a mechanism that reconciles actual and normal capacity utilisation in the long run. Yet the incorporation of a more fully developed accelerator mechanism simultaneously raises the issue of Harroddian instability. Skott (2017) challenged the robustness of this solution, arguing that, although the model may exhibit formal asymptotic stability, such stability depends on a narrow set of parameter values and is therefore better interpreted as a case of corridor stability⁵.

To contribute to this debate, the present paper introduces an explicit Keynesian quantity-adjustment mechanism for capacity utilisation into a neo-Kaleckian model with autonomous demand. This extension allows the implications of Harroddian instability to be reconsidered in a three-dimensional framework, and the Skott–Lavoie controversy to be reformulated more precisely. In particular, the loss of local stability need not imply cumulative divergence away from the steady-growth path. Instead, beyond a critical threshold, the economy may undergo a qualitative change in its dynamics, giving rise to persistent endogenous fluctuations around the long-run equilibrium.

In this sense, the contribution of the paper is not to extend the corridor of asymptotic stability of the two-dimensional model, but rather to characterise the qualitative change in the dynamics that occurs at its boundary when we extend its dimension. This interpretation is broadly consistent with Skott (2017)’s argument that Harroddian instability need not necessarily manifest itself through explosive divergence if additional stabilising mechanisms are present. In Skott (1989)’s own work, these mechanisms emerge from labour-market feedback effects, which may transform unstable growth dynamics into persistent fluctuations rather than cumulative explosions.

Such a view also dialogues with another line of criticism that has gained prominence in the recent literature. Nikiforos (2013, 2016) argues that, although the SSM may succeed in stabilising growth, it does so at the cost of eliminating endogenous business-cycle dynamics. In its basic form, the model appears “too stable,” lacking internal mechanisms that can generate the recurrent fluctuations characteristic of capitalist economies⁶. From this perspective, the SSM risks replacing Harroddian instability with an equally unsatisfactory form of excessive smoothness.

On the theoretical front, Araujo and Moreira (2025) show that the introduction of residential investment can endow the SSM with endogenous cyclical dynamics without requiring additional nonlinearities. Another related contribution is Gallo (2025), who explore cyclical dynamics in a multiplier–accelerator model with autonomous demand, whereby fluctuations emerge from the interaction between a dynamic multiplier and a nonlinear accelerator mechanism.⁷ While both contributions show that demand-led growth models may generate cyclical adjustment paths, the mechanism highlighted in the present paper is fundamentally different.

Gallo and Zamparelli (2026) also extended the SSM by incorporating induced technical change and endogenous income distribution. Their analysis shows that labour-market

⁵We use the term “corridor stability” in the sense adopted by Skott (2017), namely, to describe stability that holds only within a restricted range of parameter values, rather than in the Hicksian–Leijonhufvudian tradition, where it refers to bounds on endogenous variables that prevent explosive dynamics.

⁶This criticism, however, sits uneasily with recent empirical evidence. Summa et al. (2024) show that excluding residential investment from aggregate investment reveals a systematic lag of business investment over the cycle. Their findings document distinct cyclical roles for residential and non-residential investment, with the former leading and the latter lagging the business cycle.

⁷The emphasis in that approach is therefore on integrating the autonomous-demand-led growth literature with the traditional multiplier–accelerator theory of business cycles.

feedback and distribution-induced productivity growth may reconcile the warranted and natural growth rates while preserving demand-led growth outcomes. Besides, the resulting three-dimensional dynamical system converges to the steady state through damped oscillations, suggesting that the Harroddian instability can be effectively contained.⁸

We show that endogenous fluctuations may arise even in the absence of strong nonlinearities in the investment function. By formally characterising the conditions under which such cyclical dynamics emerge, we demonstrate that the neo-Kaleckian model with autonomous demand is neither dynamically sterile nor explosively unstable, but capable of generating bounded, demand-driven cycles while preserving a well-defined long-run anchor. In this sense, the framework navigates between the twin critiques of Harroddian instability and excessive stability.

The remainder of the paper is organised as follows. Section 2 presents the neo-Kaleckian framework with autonomous demand and its local stability properties. Section 3 investigates the model's dynamic behaviour beyond local stability by conducting a formal Hopf bifurcation analysis. Section 4 concludes by discussing the implications of these results for the debate on Harroddian instability and the alleged excessive stability of the SSM.

2 The Model

We focus on a demand-led growth model that blends the neo-Kaleckian investment function with an autonomous demand component, building on the frameworks proposed by Allain (2015) and Lavoie (2016). The economy is closed and without government. Output (Y) is demand-determined, and the technology is Leontief assuming labour surplus, which yields: $Y = uK/v$, where K is the capital stock, $v = K/Y_f$ is the capital-output ratio at full capacity and $u = Y/Y_f$ is the rate of capacity utilisation. Based on Rowthorn (1981) and Dutt (1984), and following Amadeo (1986), firms' accumulation decisions depend positively on deviations of actual capacity utilisation from its normal level, denoted by u_n , according to the investment function:

$$g_I = \gamma + \beta(u - u_n), \quad \beta > 0, \quad (1)$$

where g_I denotes the gross investment rate, which, in the absence of depreciation, coincides with the rate of capital accumulation g_K . The parameter β measures the responsiveness of investment to capacity utilisation, while γ represents firms' expected secular growth rate. In the neo-Kaleckian model, this term captures the trend component of investment, which does not depend on current utilisation and reflects firms' long-run assessment of demand expansion. Following the interpretation proposed by Allain (2015), firms revise their long-run growth expectations in light of realised growth outcomes according to:

$$\dot{\gamma} = \psi(g - \gamma), \quad (2)$$

where g is the equilibrium growth rate, $\psi \in [0, 1]$ measures the speed at which firms adjust their growth expectations toward realised growth, with $\psi = 1$ corresponding to instantaneous adjustment⁹. Saving is assumed to originate exclusively from profits, with

⁸Despite this similarity, our framework is explicitly neo-Kaleckian, with induced investment responding to deviations of capacity utilisation from its normal level, while their model remains closer to the baseline SSM augmented with endogenous technical change.

⁹As pointed out by Allain (2015), this equation introduces the possibility of Harroddian instability into the neo-Kaleckian formulation.

workers consuming their entire wage income. Aggregate demand, however, includes an autonomous component of consumption, Z , which is independent of current income. The introduction of autonomous demand represents the main departure from the canonical neo-Kaleckian framework, while the standard assumption of zero workers' saving is maintained. Under these assumptions, the saving function can be written as:

$$g_S = \frac{s\pi u}{v} - z, \quad (3)$$

where $z = Z/K$ denotes the ratio of autonomous consumption to the capital stock, π is the profit share, and s is the marginal propensity to save out of profits. Autonomous consumption is assumed to grow at the exogenously given rate g_Z . It follows that z evolves according to:

$$\dot{z} = z(g_Z - g_K). \quad (4)$$

Assuming away depreciation of the capital stock, the rate of gross investment coincides with the rate of capital accumulation: $g_I = g_K$. Moreover, according to the neo-Kaleckian view, the growth regime is investment-led: in the absence of short-run supply constraints and under conditions of excess capacity, the actual growth rate of output is determined by the pace of capital accumulation. As noted by Serrano and Freitas (2017, p. 83), “in the neo-Kaleckian model, (...) the pace of investment expansion determines the equilibrium output growth rate of the economy for a given level of income distribution.” Hence, we can assume that $g \equiv g_I = g_K$. Using this closure, equation (2) can be rewritten as:

$$\dot{\gamma} = \psi\beta(u - u_n). \quad (5)$$

This mechanism ensures that, in any steady state, capacity utilisation must coincide with its normal level. Goods-market equilibrium, however, need not hold instantaneously. Deviations between planned investment and saving are instead accommodated through variations in capacity utilisation, governed by a Keynesian quantity-adjustment mechanism. Therefore, out of equilibrium, capacity utilisation adjusts according to a standard Keynesian adjustment rule whereby excess demand in the goods market is absorbed through changes in output rather than prices¹⁰ [see e.g., Lavoie (2010), Sasaki (2013) and Nah and Lavoie (2019)] as follows:

$$\dot{u} = \theta(g_I - g_S), \quad \theta > 0, \quad (6)$$

where θ measures the speed of adjustment of the goods market. The standard 2D model [see e.g., Allain (2015), Lavoie (2016), and Skott (2017)] can be interpreted as the limiting case in which utilisation adjusts infinitely fast, so that $\theta \rightarrow \infty$ and $g_I = g_S$ holds continuously. From this perspective, it abstracts from the temporal process of adjustment insofar as the goods-market equilibrium is imposed instantaneously, and the resulting dynamic system is defined over only two state variables. By endogenising the quantity adjustment, the present framework increases the system's dimensionality and introduces an additional state variable, thereby generating richer dynamics, including the possibility

¹⁰Since output adjusts gradually, goods-market equilibrium does not hold at every instant along the adjustment path. Temporary discrepancies between demand and supply should therefore be interpreted in the standard Keynesian sense as being absorbed through mechanisms such as unplanned inventory changes, order backlogs, or unmet demand, rather than through residual adjustments in consumption behaviour.

of endogenous cycles. Then, consider the following three-dimensional system:

$$\begin{cases} \dot{\gamma} = \psi\beta(u - u_n), \\ \dot{z} = z[g_Z - \gamma - \beta(u - u_n)], \\ \dot{u} = \theta\left[\gamma + \beta(u - u_n) - \frac{s\pi u}{v} + z\right]. \end{cases} \quad (7)$$

A steady-state equilibrium (γ^*, z^*, u^*) requires $\dot{\gamma} = \dot{z} = \dot{u} = 0$. From $\dot{\gamma} = 0$, one obtains:

$$u^* = u_n. \quad (8)$$

From $\dot{z} = 0$ and $u^* = u_n$, $z(g_Z - \gamma) = 0$. If $z^* = 0$ we obtain after some algebraic manipulation:

$$\gamma^* = \frac{s\pi u_n}{v}. \quad (9)$$

Thus, the first steady-state equilibrium is given by: $E_1 : (\gamma^*, z^*, u^*) = (\frac{s\pi u_n}{v}, 0, u_n)$. We show that this equilibrium is unstable, as stated in the next proposition.

Proposition 1. *For all admissible parameters, E_1 is a locally unstable point.*

Proof. After permuting the order of variables from (γ, u, v) to (γ, v, u) , we obtain a similar matrix to the Jacobian matrix evaluated at E_1 , namely:

$$J' = PJ(E_1)P^{-1} = \begin{bmatrix} 0 & \psi\beta & 0 \\ \theta & \theta\beta - \frac{\theta s\pi}{v} & \theta \\ 0 & 0 & g_Z - \frac{s\pi u_n}{v} \end{bmatrix},$$

which is block-triangular. Hence the eigenvalues of $J(E_1)$ are the eigenvalues of the diagonal blocks: $\lambda_3 = g_Z - \frac{s\pi u_n}{v}$, and the two eigenvalues λ_1, λ_2 of the 2×2 block

$$J_{2D} = \begin{bmatrix} 0 & \psi\beta \\ \theta & \theta\beta - \frac{\theta s\pi}{v} \end{bmatrix}.$$

For J_{2D} we compute the determinant:

$$\det(J_{2D}) = 0 \cdot \left(\theta\beta - \frac{\theta s\pi}{v}\right) - \psi\beta \cdot \theta = -\theta\psi\beta.$$

Since $\theta > 0$ and $\psi\beta > 0$ by assumption, $\det(J_{2D}) < 0$. A real 2×2 matrix with negative determinant has two real eigenvalues of opposite sign. Therefore, one of λ_1, λ_2 is positive and the other negative. Consequently, $J(E_1)$ possesses at least one positive real eigenvalue, regardless of the value of λ_3 , which implies that E_1 is locally unstable. This conclusion holds for all admissible parameter values. $\square$

Besides, the equilibrium E_1 with $z^* = 0$ corresponds to a purely Kaleckian configuration without autonomous demand. We henceforth focus on the interior steady-state equilibrium characterised by $z^* \neq 0$. Now, also considering that $u^* = u_n$, from $\dot{z} = 0$:

$$\gamma^* = g_Z \quad (10)$$

and, from (4), one obtains:

$$z^* = \frac{s\pi u_n}{v} - g_Z. \quad (11)$$

The equilibrium $E_2 : (\gamma^*, z^*, u^*) = (g_Z, \frac{s\pi u_n}{v} - g_Z, u_n)$ is economically meaningful if $g_Z < \frac{s\pi u_n}{v}$. This condition ensures that autonomous demand grows more slowly than the economy's saving capacity, evaluated at normal utilisation. We can show that this non-trivial equilibrium is asymptotically stable under certain parameter constraints. In particular, local asymptotic stability requires that the real parts of all eigenvalues be negative. The following lemma derives the characteristic polynomial of $J(E_2)$ and establishes its coefficients.

Lemma 1. *The characteristic polynomial associated with $J(E_2)$ is*

$$\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 = 0, \quad (12)$$

where

$$S_1 = \theta \left(\frac{s\pi - \beta v}{v} \right), \quad S_2 = \theta\beta(z^* - \psi), \quad S_3 = \beta z^* \theta \psi, \quad (13)$$

Proof. The Jacobian matrix of system (7) evaluated at E_2 is:

$$J(E_2) = \begin{bmatrix} \theta \left(\beta - \frac{s\pi}{v} \right) & \theta & \theta \\ \psi\beta & 0 & 0 \\ -\beta z^* & -z^* & 0 \end{bmatrix} \quad (14)$$

The characteristic polynomial is obtained from $\det(\lambda I - J(E_2))$ is given by:

$$\lambda^3 + \left[\theta \left(\frac{s\pi}{v} - \beta \right) \right] \lambda^2 + \theta\beta(z^* - \psi)\lambda + \theta\beta\psi z^* = 0. \quad (15)$$

which completes the proof. $\square$

Having found the coefficients of the characteristic polynomial, we now turn to their stability implications. The following proposition focuses on the Routh–Hurwitz conditions for the asymptotic stability of the equilibrium E_2 .

Proposition 2. *Assume that $\frac{s\pi u_n}{v} > g_Z$. Then the equilibrium $E_2 = (\gamma^*, z^*, u^*)$ is locally asymptotically stable if*

$$\beta < \frac{s\pi}{v} - \frac{z^*\psi}{\theta(z^* - \psi)}. \quad (16)$$

Proof. For a third-order characteristic polynomial, the local asymptotic stability of E_2 requires the Routh–Hurwitz conditions¹¹

$$S_1 > 0, \quad S_3 > 0, \quad S_1 S_2 - S_3 > 0.$$

¹¹The condition $S_2 > 0$ is automatically satisfied when $S_1 > 0$ and $S_1 S_2 > S_3 > 0$; hence $z^* > \psi$ insofar as $S_2 = \theta\beta(z^* - \psi)$, which is one of the necessary conditions for asymptotic stability highlighted by Skott (2017) in the 2D model.

The first two conditions follow directly from the definitions of S_1 and S_3 and the positivity of the structural parameters:

$$S_1 > 0 \iff \beta < \frac{s\pi}{v} \quad (17)$$

$$S_3 > 0 \iff z^* > 0.$$

Note that (17) is nothing but the Keynesian stability condition, which also appears in the 2D version [see Skott (2017)]. Besides, $z^* > 0$ insofar as $\frac{s\pi u_n}{v} > g_Z$. Expanding $S_1 S_2 - S_3$, it yields:

$$S_1 S_2 - S_3 = \left(\frac{\theta s\pi}{v} - \theta\beta \right) \theta\beta(z^* - \psi) - \theta\psi\beta z^*. \quad (18)$$

Dividing by $\theta\beta > 0$ gives:

$$\frac{S_1 S_2 - S_3}{\theta\beta} = \theta \left(\frac{s\pi}{v} - \beta \right) (z^* - \psi) - \psi z^*. \quad (19)$$

Let $\Gamma = \theta \left(\frac{s\pi}{v} - \beta \right)$, which is positive since $S_1 > 0$. Equation (19) can then be written as $z^*(\Gamma - \psi) - \Gamma\psi$. If $\Gamma \leq \psi$, this expression is non-positive for all $z^* \geq \psi$, implying that the fourth Routh–Hurwitz condition cannot be satisfied. Hence $S_1 S_2 - S_3 > 0$ requires $\Gamma > \psi$, and the inequality $z^*(\Gamma - \psi) - \Gamma\psi > 0$ is equivalent to:

$$z^* > \frac{\Gamma\psi}{\Gamma - \psi} = \frac{\theta \left(\frac{s\pi}{v} - \beta \right) \psi}{\theta \left(\frac{s\pi}{v} - \beta \right) - \psi}. \quad (20)$$

The introduction of gradual quantity adjustment narrows the stability region because the admissible values of ψ become constrained not only by z^* , but also by the interaction between output adjustment and investment behaviour. After some algebraic manipulation, we can rewrite the inequality (20) in terms of β as in (16). $\square$

Introducing gradual quantity adjustment makes the parameter β the key determinant of local stability in the three-dimensional system. Once condition (16) holds, the standard Keynesian stability condition (17) necessarily follows. Moreover, as θ decreases, the term $\frac{z^*\psi}{\theta(z^* - \psi)}$ increases, further restricting the range of values of β compatible with local asymptotic stability. The standard Keynesian condition is recovered only in the limiting case $\theta \rightarrow \infty$, where output adjusts instantaneously, and out-of-equilibrium dynamics vanish. Hence, while a lower θ dampens the immediate output response to excess demand, gradual quantity adjustment shifts the stability boundary rather than merely attenuating multiplier–accelerator effects.

Moreover, expressing the condition in terms of β facilitates comparison with a broader literature in which the responsiveness of investment plays a central role in determining macroeconomic stability. Freitas and Serrano (2015) show that the SSM is dynamically stable provided that induced investment adjusts sufficiently gradually. Subsequent empirical work by Haluska et al. (2021) estimates the relevant adjustment parameter for the U.S. economy and concludes that the observed speed of adjustment is compatible with stability. More recently, Thompson (2024) has questioned whether the adjustment speeds

required for stability are empirically plausible, thereby shifting the discussion from the existence of a stability condition to its economic relevance¹².

Expressing the stability condition of the present model as an upper bound on β , the responsiveness of investment to deviations of utilisation from its normal level, allows us to connect directly with this debate. Consequently, the economy may lose local stability before the conventional Keynesian condition is violated, implying that the region of local asymptotic stability in the three-dimensional framework is narrower than in the corresponding two-dimensional model. In this respect, our results lend support to Skott (2017)'s interpretation in terms of corridor stability. At the same time, however, the loss of local stability need not imply explosive dynamics. As shown in the next section, crossing the stability boundary may instead trigger a Hopf bifurcation, giving rise to persistent endogenous fluctuations¹³ around the steady-growth path.

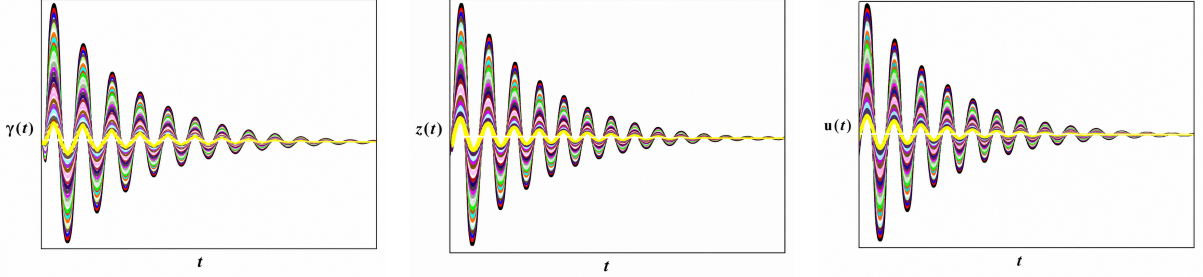


Figure 1: The interior equilibrium $E_2^* = (u^*, \gamma^*, z^*) = (0.8, 0.01, 0.086)$ is locally asymptotically stable, with eigenvalues $\lambda_1 \approx -0.0113$ and $\lambda_{2,3} \approx -0.00185 \pm 0.0129i$ when $\theta = 0.5$, $\beta = 0.097365$, $u_n = 0.8$, $\frac{s\pi}{v} = 0.12$, and $\psi = 0.01$. In this case, $S_1 = 0.0113175 > 0$, $S_2 = 0.00369987 > 0$, $S_3 = 4.1867 \times 10^{-5} > 0$, and $E = S_1 S_2 - S_3 \approx 6.35 \times 10^{-9} > 0$.

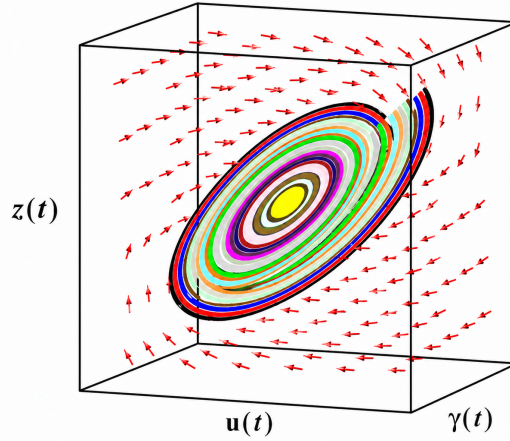


Figure 2: Phase diagrams illustrating the asymptotic stability of the equilibrium point $E_2^* = (u^*, \gamma^*, z^*) = (0.8, 0.01, 0.086)$.

¹²Thompson (2024)'s analysis is conducted in discrete time and focuses primarily on the empirical plausibility of the standard SSM stability condition derived by Summa et al. (2024)

¹³Following Skott (2019, p. 277), persistent fluctuations are not inconsistent with demand-led growth models with autonomous demand. Skott (1989) argues, however, that such dynamics would emerge from labour-market feedback effects, which are not considered in the present framework.

3 Hopf Bifurcation Analysis

Having established the stability properties of the interior equilibrium, we now investigate whether sustained cyclical patterns may emerge through a loss of stability. Among the model's parameters, β proves to be the key determinant of the transition from local stability to cyclical dynamics insofar as it governs both the Keynesian stability condition and the narrower stability requirements of the 3D system, as shown in proposition (2). As a result, changes in this parameter become central to the system's qualitative behaviour.

This interpretation is consistent with previous contributions that generated endogenous cycles within the SSM, such as Araujo and Moreira (2025) and Araujo et al. (2023), which also adopted the responsiveness of investment to deviations of capacity utilisation from its normal level as the bifurcation parameter. We therefore show that system (7) undergoes a Hopf bifurcation when β crosses a critical threshold, generating persistent oscillations in the time paths of $\gamma(t)$, $z(t)$, and $u(t)$. The analysis follows the standard Hopf bifurcation theorem (see, e.g., Guckenheimer and Holmes, 1983).

Proposition 3. *When the value of parameter β crosses a critical threshold β^* , system (7) experiences a Hopf bifurcation at the interior equilibrium E_2 . The necessary and sufficient conditions for this bifurcation to occur at $\beta = \beta^*$ are:*

1. $E(\beta^*) = S_1(\beta^*)S_2(\beta^*) - S_3(\beta^*) = 0$, where S_1, S_2, S_3 are the coefficients of the characteristic polynomial as defined in Lemma 1;
2. $\left. \frac{d}{d\beta} \Re(\lambda(\beta)) \right|_{\beta=\beta^*} \neq 0$,

where $\lambda(\beta)$ denotes a root of the characteristic equation of the Jacobian matrix evaluated at E_2 .

Proof. (1) For $S_1 > 0$, $S_2 > 0$, $S_3 > 0$ and $E(\beta) = S_1(\beta^*)S_2(\beta^*) - S_3(\beta^*) = 0$, we obtain after some algebraic manipulation that:

$$\beta^* = \frac{\pi s}{v} - \frac{\psi z^*}{\theta(z^* - \psi)} = \frac{\pi s}{v} - \frac{\psi \left(\frac{s\pi u_n}{v} - g_Z \right)}{\theta \left(\frac{s\pi u_n}{v} - g_Z - \psi \right)}. \quad (21)$$

At $\beta = \beta^*$, the characteristic equation has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$, where $\omega = \sqrt{S_2(\beta^*)} > 0$, and a third real eigenvalue $\lambda_3 = -S_1(\beta^*) < 0$.

(2) Let $\lambda = i\omega$ ($\omega > 0$) be a purely imaginary eigenvalue at $\beta = \beta^*$. Differentiating the characteristic equation with respect to β gives

$$(3\lambda^2 + 2S_1\lambda + S_2) \frac{d\lambda}{d\beta} + \lambda^2 S'_1 + \lambda S'_2 + S'_3 = 0,$$

where primes denote $d/d\beta$. Substituting $\lambda = i\omega$ and separating real and imaginary parts yields

$$\operatorname{Re} \left(\frac{d\lambda}{d\beta} \right)_{\lambda=i\omega} = - \frac{(S_2 - 3\omega^2)(S'_3 - \omega^2 S'_1) + 2S_1\omega^2 S'_2}{(S_2 - 3\omega^2)^2 + (2S_1\omega)^2}.$$

At β^* we have $\omega^2 = S_2$, so $S_2 - 3\omega^2 = -2S_2$. Hence, the numerator becomes

$$N = -2S_2(S'_3 - S_2 S'_1) + 2S_1 S_2 S'_2 = 2S_2(S_1 S'_2 + S_2 S'_1 - S'_3).$$

Since $S_2(\beta^*) > 0$, the transversality condition $\text{Re}(d\lambda/d\beta) \neq 0$ is equivalent to

$$\Delta := S_1 S'_2 + S_2 S'_1 - S'_3 \neq 0 \quad \text{at } \beta^*.$$

Now compute the derivatives explicitly.

$$S'_1 = -\theta, \quad S'_2 = \theta(z^* - \psi), \quad S'_3 = \theta z^* \psi.$$

Thus

$$\begin{aligned} S_1 S'_2 &= (\theta \frac{s\pi}{v} - \theta\beta) \theta(z^* - \psi) = \theta^2(z^* - \psi) \left(\frac{s\pi}{v} - \beta \right), \\ S_2 S'_1 &= \theta\beta(z^* - \psi) (-\theta) = -\theta^2\beta(z^* - \psi), \\ S'_3 &= \theta z^* \psi. \end{aligned}$$

Therefore

$$\Delta = \theta^2(z^* - \psi) \left(\frac{s\pi}{v} - \beta - \beta \right) - \theta z^* \psi = \theta^2(z^* - \psi) \left(\frac{s\pi}{v} - 2\beta \right) - \theta z^* \psi.$$

At the Hopf point β^* we have the condition $S_3 = S_1 S_2$:

$$\beta z^* \theta \psi = \left(\theta \frac{s\pi}{v} - \theta\beta \right) \theta\beta(z^* - \psi).$$

Cancelling $\theta\beta$ (assuming $\beta^* \neq 0$) gives

$$z^* \psi = \theta \left(\frac{s\pi}{v} - \beta^* \right) (z^* - \psi).$$

Substituting this into the expression for Δ at β^* :

$$\Delta(\beta^*) = \theta^2(z^* - \psi) \left(\frac{s\pi}{v} - 2\beta^* \right) - \theta z^* \psi.$$

Write $\frac{s\pi}{v} - 2\beta^* = \left(\frac{s\pi}{v} - \beta^* \right) - \beta^*$. Then

$$\Delta(\beta^*) = \theta^2(z^* - \psi) \left(\frac{s\pi}{v} - \beta^* \right) - \theta^2(z^* - \psi) \beta^* - \theta z^* \psi.$$

But $\theta^2(z^* - \psi) \left(\frac{s\pi}{v} - \beta^* \right) = \theta \cdot \left[\theta \left(\frac{s\pi}{v} - \beta^* \right) (z^* - \psi) \right] = \theta(z^* \psi)$ from the Hopf condition. Hence the first and third terms cancel:

$$\Delta(\beta^*) = \theta z^* \psi - \theta^2(z^* - \psi) \beta^* - \theta z^* \psi = -\theta^2 \beta^* (z^* - \psi).$$

Thus

$$\Delta(\beta^*) = -\theta^2 \beta^* (z^* - \psi).$$

For a non-degenerate Hopf bifurcation, we require $\Delta(\beta^*) \neq 0$. This holds provided $\beta^* \neq 0$ and $z^* \neq \psi$. Therefore, under the generic assumptions of the problem, $\Delta(\beta^*) \neq 0$. Consequently:

$$\text{Re} \left(\frac{d\lambda}{d\beta} \right) \Big|_{\lambda=i\omega} \neq 0,$$

which is exactly the transversality condition required for a non-degenerate Hopf bifurcation. This completes the proof. □

Proposition (3) clarifies the precise manner in which local stability is lost in the three-dimensional system. As discussed in the previous section, the introduction of gradual quantity adjustment narrows the region of local asymptotic stability relative to the conventional two-dimensional framework. However, crossing this tighter stability boundary does not necessarily imply cumulative or explosive instability. Instead, when β reaches the critical threshold β^* , the system undergoes a Hopf bifurcation in which a pair of complex conjugate eigenvalues crosses the imaginary axis while the remaining eigenvalue is negative. The loss of local asymptotic stability, therefore, marks a qualitative transition from convergence toward the steady-growth path to persistent oscillatory dynamics around it¹⁴.

This result is important because it refines the interpretation of corridor stability advanced by Skott (2017). Although the three-dimensional framework indeed implies a narrower stability corridor, the economy does not necessarily diverge explosively once this boundary is crossed. Rather, the destabilising effects associated with Harroddian adjustment may manifest themselves through the emergence of persistent business-cycle-type fluctuations. In this sense, the loss of local stability gives rise to bounded endogenous cycles rather than cumulative divergence, as shown in the figures below¹⁵.

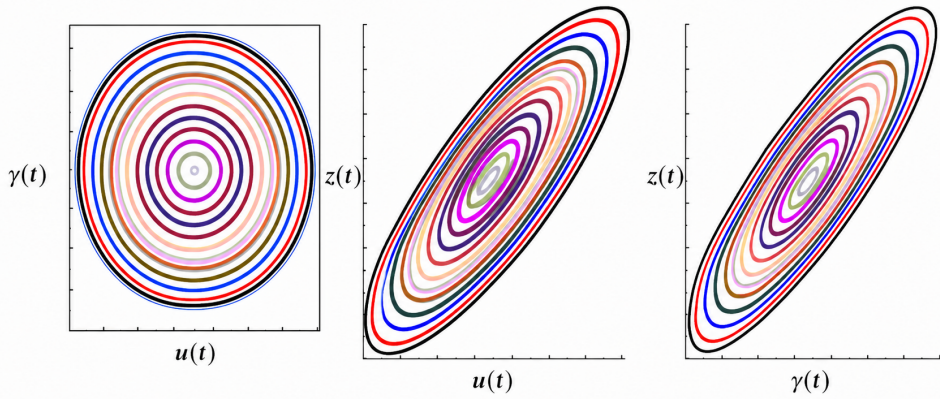


Figure 3: Endogenous cycles when $\theta = 0.5$, $\beta = 0.097369$, $u_n = 0.8$, $\frac{s\pi}{v} = 0.12$, and $\psi = 0.01$. In this case, $S_1 \approx 0.011315 > 0$, $S_2 \approx 0.003700 > 0$, $S_3 \approx 4.1869 \times 10^{-5} > 0$, and $E = S_1 S_2 - S_3 \approx -1.35 \times 10^{-9} < 0$. The interior equilibrium E_2^* becomes locally unstable (a saddle-focus with a two-dimensional unstable manifold), with eigenvalues $\lambda_1 \approx -0.0113$ and $\lambda_{2,3} \approx +1.2 \times 10^{-8} \pm 0.0608i$. Because the Hopf bifurcation is supercritical, a stable limit cycle emerges, giving rise to persistent endogenous cycles.

¹⁴As shown by Skott (2017), the sign of the real part of the eigenvalues in the two-dimensional model depends exclusively on $\psi - z^*$ and is therefore independent of β . Consequently, unlike in the three-dimensional model, variations in β cannot induce a Hopf bifurcation, since they cannot cause the real part of the eigenvalues to cross zero.

¹⁵In Skott (1989), such mechanisms arise from labour-market feedback effects capable of transforming unstable growth dynamics into persistent fluctuations rather than cumulative explosions

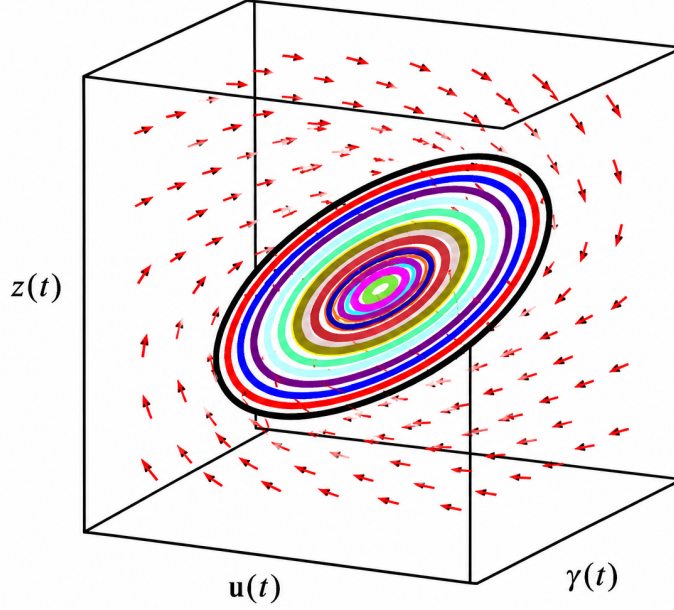


Figure 4: Endogenous cycle following the Hopf bifurcation, exhibiting stable and bounded oscillations.

To illustrate the emergence of endogenous cycles in the system (7), consider that $\theta = 0.5$, $u_n = 0.8$, $s\pi/v = 0.12$, $\psi = 0.01$, and $g_Z = 0.01$. The corresponding interior equilibrium is $E_2 = (u^*, \gamma^*, z^*) = (0.8, 0.01, 0.086)$. Moreover

$$S_1(\beta) = 0.06 - 0.5\beta, \quad S_2(\beta) = 0.038\beta, \quad S_3(\beta) = 0.00043\beta,$$

and

$$E(\beta) = S_1(\beta)S_2(\beta) - S_3(\beta) = 0.00185\beta - 0.019\beta^2.$$

Solving $E(\beta) = 0$ (for $\beta > 0$) yields the critical value $\beta = \beta^* = 0.097368421$, which matches β_H . Linearising the system around E_2 , the Jacobian matrix at $\beta = \beta^*$ has eigenvalues

$$\lambda_1 = -S_1(\beta^*) \approx -0.0113158$$

and

$$\lambda_{2,3} = \pm i\sqrt{S_2(\beta^*)} \approx \pm 0.060827i,$$

indicating the presence of purely imaginary eigenvalues. Moreover,

$$\left. \frac{d \operatorname{Re}(\lambda_j(\beta))}{d\beta} \right|_{\beta=\beta^*} = \frac{S'_3(\beta^*) - S'_1(\beta^*)S_2(\beta^*) - S_1(\beta^*)S'_2(\beta^*)}{2[S_1^2(\beta^*) + S_2(\beta^*)]} \approx 0.2416 \neq 0,$$

so a Hopf bifurcation occurs at $\beta = \beta^*$.

Hence, all the conditions of the Hopf bifurcation theorem are satisfied for system (7). Moreover, the analysis presented in the Appendix shows that the Hopf bifurcation occurring at the critical value β^* is supercritical. Consequently, there exists a neighbourhood of the bifurcation point, $(\beta^*, \beta^* + \varepsilon)$, in which a locally asymptotically stable limit cycle emerges.¹⁶ This result has important economic implications.

¹⁶Hopf bifurcation theory does not determine the global extent of this interval. This depends on the higher-order terms of the vector field and on the global properties of the nonlinear dynamics.

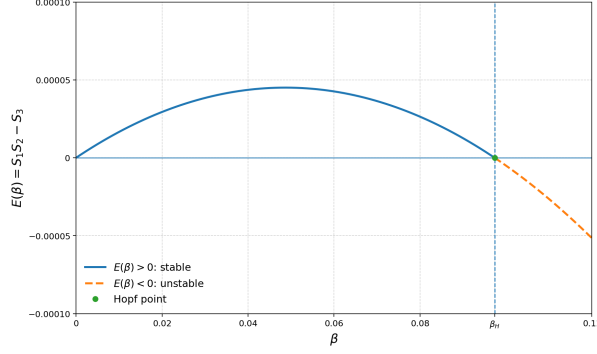


Figure 5: The bifurcation condition is given by $E(\beta) = 0$, where $E(\beta) = 0.00185\beta - 0.019\beta^2$ in the β -axis. The dotted vertical line at $\beta = \beta_H \approx 0.097368421$ identifies the Hopf bifurcation point where the qualitative dynamics change. The shaded regions correspond to $E(\beta) > 0$ (for $0 < \beta < \beta_H$) and $E(\beta) < 0$ (for $\beta > \beta_H$), indicating local asymptotic stability and instability of the interior equilibrium, respectively.

Although the steady state loses local stability at β^* , the ensuing fluctuations remain bounded and non-explosive, with the economy converging to a stable periodic orbit rather than exhibiting cumulative instability. In a Harrodian context, this implies that even when the system exits the corridor of stability, destabilising forces may not lead to cumulative instability. Instead, the economy may converge to a regular, bounded cycle—effectively preventing the explosive knife-edge behaviour often associated with Harrodian instability, while still capturing persistent cyclical dynamics outside the stable equilibrium region.

4 Concluding Remarks

This paper has examined the dynamic properties of an extended neo-Kaleckian model with autonomous demand, building on the contributions of Allain (2015) and Lavoie (2016). Unlike those two-dimensional frameworks, we introduce a Keynesian quantity-adjustment mechanism for capacity utilisation, thereby increasing the system's dimensionality and allowing explicit out-of-equilibrium dynamics. Our analysis shows that the interior equilibrium remains locally asymptotically stable over a non-trivial region of the parameter space. However, as the responsiveness of investment to capacity utilisation approaches a critical threshold, stability is lost through a Hopf bifurcation, giving rise to endogenous cyclical fluctuations.

The resulting dynamics are characterised not by cumulative instability but by stable oscillations in capacity utilisation, growth expectations, and the ratio of autonomous demand to the capital stock. These findings speak directly to the concerns raised by Skott (2017) regarding the restrictive parameter configurations required for stability in autonomous-demand-led growth models, which he interprets as being, at best, consistent with corridor stability. While we agree that asymptotic stability relies on relatively stringent parameter restrictions, our results show that moving beyond this stability corridor does not necessarily imply the emergence of Harrodian instability.

Instead, crossing the stability boundary marks a qualitative change in the economy's behaviour. The system transitions from convergence towards the steady-growth path to persistent endogenous fluctuations around it, while preserving the long-run anchoring

role of autonomous demand. In this sense, the loss of local stability does not lead to explosive dynamics but to a bounded cyclical regime arising from the interaction among induced investment, autonomous demand, and gradual output adjustment. These results suggest that neo-Kaleckian models with autonomous demand are neither dynamically sterile nor inherently unstable. Once out-of-equilibrium adjustment processes are explicitly incorporated, they become capable of generating persistent endogenous cycles around a well-defined long-run growth path.

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Appendix

Direction and Stability of the Hopf Bifurcation

We apply normal form theory to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions of system (7), following Hassard et al. (1981, pp. 86–91). The parameters are

$$\theta = 0.5, \quad u_n = 0.8, \quad \frac{s\pi}{v} = 0.12, \quad \psi = 0.01, \quad g_Z = 0.01.$$

The Hopf bifurcation occurs at

$$\beta_H = \frac{s\pi}{v} - \frac{\psi z^*}{\theta(z^* - \psi)} = 0.0973684210526,$$

where the interior equilibrium is

$$E_2 = (u^*, \gamma^*, z^*) = (0.8, 0.01, 0.086).$$

We translate the equilibrium point to the origin by defining

$$u_1 = u - 0.8, \quad \gamma_1 = \gamma - 0.01, \quad z_1 = z - 0.086.$$

The translated system can be written as

$$X' = AX + B,$$

where

$$X = \begin{pmatrix} u_1 \\ \gamma_1 \\ z_1 \end{pmatrix},$$

and

$$A = J(E_2) = \begin{pmatrix} -0.011315789474 & 0.5 & 0.5 \\ 0.000973684211 & 0 & 0 \\ -0.008373684211 & -0.086 & 0 \end{pmatrix}.$$

The nonlinear terms are

$$B = \begin{pmatrix} 0 \\ 0 \\ -\gamma_1 z_1 - \beta_H u_1 z_1 \end{pmatrix}.$$

The eigenvalues of A at $\beta = \beta_H$ are

$$\lambda_{1,2} = \pm 0.060827625303 i, \quad \lambda_3 = -0.011315789474.$$

Thus,

$$\omega_0 = 0.060827625303.$$

We choose the nonsingular matrix

$$P = \begin{pmatrix} -0.990283769480 & 0 & -0.992677224979 \\ 0 & -0.015851739493 & 0.085416412382 \\ -0.022411685309 & 0.136324959640 & -0.085416412382 \end{pmatrix},$$

with inverse

$$P^{-1} = \begin{pmatrix} -0.976033651069 & -12.835606122515 & -1.492512339827 \\ -0.181571962734 & 5.912786542589 & 8.022947190581 \\ -0.033696468539 & 12.804658044651 & 1.488913726122 \end{pmatrix}.$$

This transformation reduces the linear part to the real canonical form

$$P^{-1}AP = \begin{pmatrix} 0 & -0.060827625303 & 0 \\ 0.060827625303 & 0 & 0 \\ 0 & 0 & -0.011315789474 \end{pmatrix}.$$

Let $X = PY$. Then

$$Y' = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} Y + F(Y),$$

where $Y = (y_1, y_2, y_3)'$ and

$$F(Y) = \begin{pmatrix} F_1(Y) \\ F_2(Y) \\ F_3(Y) \end{pmatrix}.$$

The nonlinear terms are

$$F_1(Y) = 0.003225300878y_1^2 - 0.019088503132y_1y_2 + 0.012668351656y_1y_3 \\ - 0.003225300878y_2^2 - 0.000265902604y_2y_3 + 0.001432804679y_3^2,$$

$$F_2(Y) = -0.017337490571y_1^2 + 0.102609572120y_1y_2 - 0.068098275384y_1y_3 \\ + 0.017337490571y_2^2 + 0.001429350027y_2y_3 - 0.007701990776y_3^2,$$

$$F_3(Y) = -0.003217524318y_1^2 + 0.019042478622y_1y_2 - 0.012637806848y_1y_3 \\ + 0.003217524318y_2^2 + 0.000265261484y_2y_3 - 0.001429350027y_3^2.$$

Following the algorithm in Hassard et al. (1981, pp. 86–91), the relevant normal-form coefficients are

$$g_{11} \simeq 0,$$

$$g_{02} = -0.048079485182 - 0.026881742136 i,$$

$$g_{20} = 0.054530086938 - 0.007793239005 i.$$

Moreover,

$$h_{20} = -0.003217524318 - 0.009521239311 i,$$

and

$$w_{20} = 0.080031739915 - 0.019003717360 i.$$

The remaining coefficients are

$$G_{110} = 0.007048850842 - 0.033916186390 i,$$

$$G_{101} = 0.005619500815 - 0.034182088994 i.$$

Since $G_{21} = 0$ and $w_{11} = 0$, we obtain

$$g_{21} = G_{101}w_{20} = -0.000199848330 - 0.002842443461 i.$$

The first Lyapunov coefficient is then

$$c_1(0) = \frac{1}{2} \frac{i}{\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21}.$$

Substitution yields

$$c_1(0) = -0.000099924165 - 0.009735056496 i.$$

Since

$$\operatorname{Re}(c_1(0)) = -0.000099924165 < 0,$$

the Hopf bifurcation at $\beta = \beta_H$ is supercritical. Therefore, the bifurcating limit cycle is locally asymptotically stable.