

Stagnation and cycles in Marx's Circuit of Capital

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Abstract

Marx' circuit of capital describes the circular process of transformation of money to commodities and of commodities to money, the unified process of capital turnover from production to exchange. It becomes the prime tool Marx used to analyze the process of labor, the reproduction and accumulation of capital, and the possibility and actuality of crises in the capitalist mode of production. In a paper in 1982 and two books in 1986, Foley formulated a mathematical model for the circuit of capital. In the present work, we reformulate this model into a closed and autonomous dynamical system and we proceed to analyze its phase space; not surprisingly, this model resembles the famous epidemiological models, used to describe a similar circular process for the spread of a disease. The equilibrium points of the system reveal the cases for a 'normal' phase of expansion, as well as for an 'excess capital' crisis; the shift of stability from the one to the other reveals a possibility theory of crisis as a secular stagnation process, while the shift from stability to cycles reveals an actuality theory of short-run fluctuations due to 'excess commodities' and 'excess money'.

Keywords: circuit of capital, capital accumulation, economic crisis, financialization

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1. Introduction

Setting out their theories for the capitalist economy, the Physiocrats, the Classical political economists and Marx approached the economic phenomena as circular processes, as flows of commodities and money in unity. This was proved a fundamental concept of their theories, that runs from the formation of values and prices to the outburst of crises. Based on these circular processes, Quesnay developed the Economic Table – itself depicting a circular flow of commodities and money; Smith and Ricardo developed the labor theory of value, realizing that it is labor that determines the production and exchange of commodities; and Marx developed a unified theory of money and commodities, the Law of Value, the Law of Capital Accumulation and finally the Schemes of Reproduction. In these circular processes, it became apparent that production and exchange must go hand by hand to ensure the reproduction of the system, as commodities exit the production from the one side as objects of labor (products) and enter from the other as means of labor (raw materials, tools, engines).

It is remarkable that, despite this simple fact, the circulation of money and commodities is seldom treated, even by those who originally perceived and proposed it. Indeed, Quesnay remained largely on a phenomenological approach of the dual circulation of money and commodities; though he realized that the exchange is dual in essence (money turns to commodities and commodities turn to money at the same moment), he did not explore this relation deeper to uncover the nature and role of money. The Classical political economists, such as Smith and Ricardo, adopted simple explanations for the latter and emphasized on the production and exchange mostly from the side of the commodities; without discarding the role of money, they would not incorporate it into their overall picture of the capitalist laws of motion. It was Marx who delved deeper into this dual relation, coming up with a powerful analytical tool.

In this paper, we shall deal with Marx's circuit of capital as set out in *Capital II*, and Foley's (1982 1986a and 1986b) mathematical interpretation of it. The scope is to reformulate his equations as a closed dynamical system and analyze its phase space; it appears that this dynamical system resembles the famous SIRS epidemiological models, hence leads to the existence of limit cycles and chaotic phenomena. We intend to explore these turbulent patterns by means of bifurcations and with respect to Marx's two types of crises, the 'excess capital crisis' and the 'excess commodity crisis' and the possibility of financialization.

Section 2 attempts a presentation of Marx's circuit of capital, while Section 3 contains Foley's original system; then, this system is rewritten with respect to new variables, so as to be autonomous, closed and consistent. This new model is analyzed in Section 4, where the equilibrium solutions and their stability are addressed; through this, the specific cases of 'excess capital', 'excess commodity' and 'excess money' are considered. Section 5 concludes the paper.

2. The Circuit of Capital

In Marx's perspective, commodities (C) are sold for money (M), which then buys other commodities (C) of the same value; thus, the circuit of the commodities –from the standpoint of the seller, entering the process with a commodity– can be simply described as

$$C - M - C$$

In the same manner, money is used to buy commodities, that can be sold to acquire money; thus, the circuit of money – from the standpoint of the buyer, entering the process with money– can be described as

$$M - C - M$$

The two processes appear as different links in the same chain: where the one link stops, the other should start and *vice versa*. And indeed, they do appear together, inextricably linked at all times.

In each phase the same material elements confront each other, namely a commodity and money, and the same economic dramatis personae, a buyer and a seller. Each circular path is the unity of the same two antithetical phases, and in each case this unity is mediated through the emergence of three participants in a contract, of whom one only sells, another only buys and the third both buys and sells. (*Capital* I: 248-249)

However, they are fundamentally different, and it is this difference that brings into light the very nature of the capitalist mode of production.

On the one hand, in the case of the first process, the circuit of commodities, the initial and the final state differ only in quality, as the seller enters with one type of commodity and exits with another, whose value however should be equal. This difference in quality is in fact imposed by the process, as someone entering the circuit of commodities would only do so to acquire a good he wants but

does not have. Similarly, the conservation of value throughout is also imposed by the process, as both types of commodities have been (fairly) exchanged for the same amount of money.

On the other hand, in the case of the second process, the circuit of money, the initial and the final state are of the same quality (money) but may –and should– differ in quantity. Once again, this is imposed by the process, as someone entering the circuit of money would never do so without a motive – and the exchange of money for money has a motive only as long as more money appears in the end.

In the circulation C-M-C, the money is in the end converted into a commodity which serves as a use-value; it has therefore been spent once and for all. In the inverted form M-C-M, on the contrary, the buyer lays out money in order that, as a seller, he may recover money. By the purchase of his commodity he throws money into circulation, in order to withdraw it again by the sale of the same commodity. He releases the money, but only with the cunning intention of getting it back again. The money therefore is not spent, it is merely advanced. (*Capital I*: 249)

In short, the aim of the circuit of commodities is to satisfy one's needs of goods by allowing for the acquisition of many different types of goods; but, the aim of the parallel circuit of money is to satisfy one's avarice, to extort more money out of a given sum of them. Thus, Marx proceeded to write the circuit again as

$$M - C - M'$$

where $M' > M$.

Then, the question arises: how does more money appear in the end of the process if in the parallel circuit of money everything is fairly exchanged and the initial and final state are of equal value? And to answer this, Marx went even further to ask what type of commodity may stand between two sums of money? To answer this, one has to look behind the process of circulation and into the process of production. Since no commodity can exist –and thus be exchanged– unless it is produced, the process of circulation of money must cease temporarily for the process of production to take place. This means that the commodity intervening between two sums of money is not a unique commodity, but rather it enclosed a process where commodities through the mediation of labor power turn to other commodities; hence a process in the form

$$C \dots P \dots C'$$

where C and C' are two different types of commodities and P symbolizes the process of production. Marx argued that the process of production is

[...] on the one hand [...] the unity of the labour-process and the process of creating value, [thus the] production of commodities; [and] on the other hand [...] the unity of the labour-process and the process of producing surplus-value, [thus] it is the capitalist process of production, or capitalist production of commodities (*Capital I*: 304).

That is, the transformation of commodities (raw materials, tools, engines) to other commodities (products) through the use of labor power, that is dominated by capital through its property relations over both the initial and the final commodities. Consequently, labor power is itself considered a commodity and must enter the production process in the same manner – purchased by the capitalist along the means of production. Given that labor power, as every other commodity, should be exchanged at its value, that is “determined, as in the case of every other commodity, by the labour-time necessary for the production, and consequently also the reproduction, of this special article” (*Capital I*: 274), any value added to the object of labor that is beyond the value of labor power is regarded as a surplus rightfully owned by the owner of the final product – the capitalist.¹ As a result, C and C' do not differ only in quality (being different types of commodities) but also in quantity, with C' being of greater value than C.

Putting this into the context of the circuit of money, we must admit that money in this form matters mainly as a face of capital – money capital. We kick off with a capitalist who advances a sum of money in order to exchange commodities that will be used in the production process; these commodities are distinguished in two categories – the means of production (m_p) and labor power (ℓ_p). The combination of these two categories of commodities under the given social and productive relations forms the productive capital (P). Hence, the first half of the circuit of money capital is of the form

¹ From this, Marx proceeded to name the means of labour, or means of production purchased by the capitalist, *constant capital*, as their value does not alter throughout the process of production and it is merely transferred to the final product; the labour-power purchased by the capitalist, *variable capital*, as its value depends on the extent and intensity of the labour-time on which it is consumed, hence the value it transfers to the final product may differ significantly from its own exchange value; and the residue of unpaid labour-power used in the production process, that was transferred to the final product, *surplus value*.

$$M - C \left\langle \begin{matrix} m_p \\ \ell_p \end{matrix} \dots P \right.$$

Moving on to the next phase, as the process of production finishes, the capitalist now is left –apart from engines and stocks of unused raw materials– with a sum of finished commodities intended for selling. These commodities, if sold entirely, will bring about a new sum of money equal to the old sum (M) plus an increment (ΔM), which is the realized surplus value. Now, the second half of the circuit is written as

$$P \dots C' - \underbrace{M'}_{M + \Delta M}$$

Of course, the entire circuit of money capital is

$$M - C \left\langle \begin{matrix} m_p \\ \ell_p \end{matrix} \dots P \dots C' - \underbrace{M'}_{M + \Delta M} \right.$$

This formula describes the flow of value of the (individual or social) capital, appearing in the form of money – that is, the value of (individual or social) capital that enters and exits the process of circulation in the form of money. Marx (*Capital* I, chs. 3-4, *Capital* II, ch. 1) begun his analysis in this form, as it is in this form that commodities, money, and eventually capital appear in everyday transactions. However, he went on to develop similar formula –and discuss the similar relations– for the other two major forms of capital: productive capital (*Capital* II, ch. 2) and commercial capital (*Capital* II, ch. 3).

In the case of the circuit of productive capital, the general formula is

$$P \dots C' - M' - C \dots P$$

and it

[...] signifies the periodical renewal of the functioning of productive capital, hence its reproduction, or its process of production as a process of reproduction aiming at the self-expansion of value; not only production but a periodical reproduction of surplus-value; the function of industrial capital in its productive form, and this function performed not once but periodically repeated, so that the renewal is determined by the starting-point. (*Capital* II: 144)

Thus, two things are important to stress out. First of all, unlike the circuit of money capital or the circuit of commercial capital, the circuit of productive capital is not a process of circulation interrupted by the process of production, but the process of production renewed through the process of circulation. Second, the renewal of the process of production implies the process of reproduction, which brings forth the problem of capitalization of surplus value and the distinction between Simple and Extended Reproduction; thus, the circulation of productive capital brings forth the question of how the realized new value (and surplus value) of M' returns to production by being exchanged anew for means of production and labor power. Since $M' = M + \Delta M$, the question can be rephrased as ‘how and in which proportions M and ΔM after being realized are capitalized and hence reenter the process of production?’. Given the two extreme cases:

- 1) of Simple Reproduction, where surplus value is treated as personal income for the capitalist and aimed to either personal consumption, then the circuit of productive capital is

$$P \dots C' - \left\{ \begin{array}{l} M - C \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \dots P \\ \Delta M - c' \end{array} \right.$$

where P is of the same value at the initial and final state and c' are commodities of different type (and value) from C , intended for the personal consumption of the capitalist.

- 2) of Extended Reproduction, where surplus value is treated as capital and aimed (entirely) to expand the productive capital, then the circuit of the latter becomes

$$P \dots C' - \left\{ \begin{array}{l} M - C \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \\ \Delta M - c \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \end{array} \right\} \dots P$$

where P in the final state is of greater value than P in the initial state and C and c are commodities of the same type (and different value), both intended for production.

Of course, neither of these cases are of practical importance as neither occurs systematically in reality; what is of practical importance is all the intermediate cases, where a (smaller or larger) portion of the realized surplus value (ΔM) is capitalized and P in the final state is –by this fraction– of greater value than P in the initial state. Another case of practical importance is when a portion of the realized surplus value (ΔM) is treated as neither income or capital, but as hoard; in this case an amount of money is temporarily withdrawn from the circuit of industrial capital but may be advanced at a later stage. Following this, the case of credit and borrowing is also interesting, as a

portion of ΔM may be indirectly supplied to production. Consequently, a more complex form of the circuit of productive capital could be

$$P \dots C' - M' \left\{ \begin{array}{l} M - C \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \\ s \Delta M - c \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \\ q \Delta M \end{array} \right\} \dots P \dots C' - \left\{ \begin{array}{l} M \\ \Delta M \end{array} \right. \underbrace{\hspace{1cm}}_{\text{after some periods}} - C \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \dots P \left. \vphantom{\left\{ \begin{array}{l} M - C \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \\ s \Delta M - c \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \\ q \Delta M \end{array} \right\}} \right\} - C' \dots P$$

$$(1 - s - q)\Delta M - c'$$

where s the propensity to invest and q the propensity to hoard (real-valued positive parameters such that $0 \leq s + q \leq 1$); also, where c' is of different type from both C and c , as it denotes commodities used for personal consumption rather than for investment.²

In the case of the circuit of commercial capital (or commodity capital), it resembles the circulation of commodities in the same manner that the circulation of money capital resembles the circulation of money. The circulation of money as a mere mean of exchange becomes the circulation of money as the bearer of the social and productive relations of capitalism – as the face by which capital enters the transactions; similarly, the circulation of commodities that reflects the transcendence from exchange to use value, becomes the circulation of commodities that reflects the pursue of exchange value alone – and thus the pursue of profit. The general formula of this circuit is

$$C' - M' - C \left\langle \begin{array}{l} m_p \\ \ell_p \end{array} \right\rangle \dots P \dots C'$$

As Marx notes

² One should also notice other possible uses of realized surplus value, used to finance consumption at later periods by means of borrowing or withdrawing from the hoard; or the case where hoarding and/or personal consumption exceed the realized surplus value with the capitalist re-tying a smaller amount of money capital in production (Mandel 1969, ch. 10). However, these cases are not as characteristic to the capitalist mode of production. The latter is encountered only sporadically, in the cases of a deep depression, while the former can be regarded as a side effect of capitalist accumulation in the cases prior to the outburst of a crisis (Tsoulfidis and Talikis 2019, chs. 8-9).

The commodity-capital circuit, on the contrary, does not open with just capital-value but with capital-value augmented in the commodity-form. Hence it includes from the start the circuit of not only capital-value existing in the form of commodities, but also of surplus-value. Consequently if simple reproduction takes place in this form, the C' at the terminal point is equal in size to the C' at the starting-point. If a part of the surplus-value enters into the capital circuit, C'' , an enlarged C' , appears at the close instead of C' . This is merely a larger C' than that of the proceeding circuit, with a larger accumulated capital-value. Hence it begins its new circuit with a relatively larger, newly created surplus-value. In any event C' always inaugurates the circuit as a commodity-capital which is equal to capital-value plus surplus-value. (*Capital* II: 168)

In short, commercial capital appears directly different from the other two forms of capital; both money and productive capital begin their circulation without any presently generated surplus value, while the commercial capital contains presently generated surplus value that has not been realized yet. Consequently, the commercial capital, though a fundamental form of capital that is immediately perceived in everyday transaction, appears as a secondary form of capital in the overall process of circulation, dominated by the process of production – the commercial capital cannot come into existence without the productive capital, hence the circuit of commercial capital cannot be initiated without the pre-initiated circuit of productive capital.

Apart from the afore-mentioned similarities and differences between the three types of capital and their respective circuits, another important revelation from Marx's presentation is the links and the time intervals between different phases in the process of circulation. Namely, the circulation of money capital gives its place to the circulation of productive capital as the capital loses its money-form and takes its industrial- or productive-form; similarly, the circulation of productive capital gives its place to the circulation of commercial capital as the capital loses its industrial-form and takes its commodity-form; finally, the circulation of commercial capital reverses to the circulation of money capital as the commodity-form of capital is realized to the money-form. Each of these transformations is accompanied by a time interval. Namely,

1. the transformation of money capital to productive capital has a time lag for the recommittal of money to production, related to the conditions of the financial markets; if the financial

markets are tight due to a generalized overproduction, or due to inflationary pressures, the recommittal will be delayed and hoarding may be preferred.

2. the transformation of productive capital to commercial capital has a time lag for the process of production to take place; this is mainly related to labor productivity, but also to the specifics of the technical conditions of production of different commodities.
3. the transformation of commercial capital to money capital has a time lag for the realization, that is to the transformation of value from the commodity-form to the money-form, related to the conditions of commodity markets; if the commodity markets face either an excess supply (generalized overproduction) or a lack of demand (generalized underconsumption), then the transformation will be delayed.

These time intervals, especially the first and second one, are important in driving the conditions for two types of crises, present in Marx's theory, as well as the conditions for the contemporary theory of financialization. As Itoh (1978) noted, Marx has two different theories of crises, namely 'type I' and 'type II'. The first type, also coined theory of 'excess capital crisis' or 'overaccumulation crisis', is brought about because of over-accumulated money capital that cannot be, or should not be transformed into productive capital, due to strict economic conditions. This is related to the Law of the Falling Rate of Profit, as the accumulation of capital (increase in the composition of capital) cannot be halted by the production of surplus value (increase in the rate of surplus value), that is the extension and intensity of the working day, and leads to a slowdown and eventually a generalized breakdown of the economy.

There would be absolute over-production of capital as soon as additional capital for purposes of capitalist production = 0. The purpose of capitalist production, however, is self-expansion of capital, i.e., appropriation of surplus-labour, production of surplus-value, of profit. As soon as capital would, therefore, have grown in such a ratio to the labouring population that neither the absolute working-time supplied by this population, nor the relative surplus working-time, could be expanded any further (this last would not be feasible at any rate in the case when the demand for labour were so strong that there were a tendency for wages to rise); at a point, therefore, when the increased capital produced just as much, or even less, surplus-value than it did before its increase, there would be absolute over-production of capital; i.e., the increased capital $C + \Delta C$ would produce no more, or even less, profit than capital C before its expansion by ΔC . (*Capital* III: 360)

In short, despite the existence of a great amount of labor power and means of production, despite the existence of a large sum of money intended to be recommitted to production, the conversion from the money-form to the productive-form is not possible or preferable to capitalists, since new capital will not yield as much surplus value as it is normally anticipated to.

The second type, known as theory of ‘excess commodity crisis’ is related to the inability of realization, hence to the inability of the commodity-form of capital to transform into the money-form. As mentioned, this is due to two main causes: excess supply (overproduction) of commodities, or lack of demand for commodities (underconsumption). As for the first, overproduction of commodities is possible “as the result of a disproportion of production in various branches of the economy, and as a result of a disproportion between the consumption of the capitalists and their accumulation” (*Capital* III: 347); this means that the excess growth of particular industries, accompanied with excess investment to capital stock –and hence excess increase in productive capacity and in labor productivity– may lead to the production of more commodities than actually needed.³ As for the second, it comes to the ability to sell this commodities and realize the respective value and surplus value they embody; however, “the consuming power of the workers is limited partly by the laws of wages, partly by the fact that they are used only as long as they can be profitably employed” (*Capital* III: 347), hence the realization becomes impossible.⁴

Of course, as Itoh (1978) and Clarke (2016, ch. 9) mention, both tendencies co-exist and both are present during a crisis. Essentially, if excess (money) capital exists due to overaccumulation, then it is apparent that an excess amount of commodities (raw materials, tools, engines) exist and are unsold, as money capital does not transform into productive; similarly, if excess commodities exist due to overproduction (and underconsumption), then realization of commodity capital to money capital is impossible and this becomes a signal for existing money capital to stagnate and do not

³ We should note here that this is strongly related to the disproportions theory of crisis by Tugan-Baranovskii and others, whereas the imbalances between different economic sectors caused by the process of capital accumulation are deepened over time and cannot be mended but for an overproduction crisis (Trigg 2006). It also resembles the Harrod-Domar theory of secular stagnation, due to capitalists’ misleading to invest further when they already over-produce.

⁴ This is reflected to later underconsumption theories by Luxemburg, Moszkowska and others, who stated the “The ultimate reason for all real crises always remains the poverty and restricted consumption of the masses as opposed to the drive of capitalist production to develop the productive forces as though only the absolute consuming power of society constituted their limit” (*Capital* III: 347). But, it is also related to many post-Keynesian theories of crises (Clarke 2016).

transform further into productive capital. Indeed, the emergence of one type of crisis leads directly to the other, both of them intermingles in a generalized breakdown. However, since these two tendencies are logically opposed, we must figure which is the dominant one – which one generates the conditions for the other. While Clarke is reluctant to identify one tendency as the predominant condition for crises, Itoh seems convinced that overproduction of commodities is the main form by which a crisis appears in the capitalist system. However, as discussed by Shaikh (2016, ch. 16) and Tsoulfidis and Tsaliki (2019, ch. 8) argue that this is merely the ‘actuality’ of the crisis, that is the way by which a crisis surfaces; the ‘possibility’ of it, that is the general condition of a crisis lies with the declining profitability in the long run and the subsequent overaccumulation.

3. A Mathematical Interpretation

Foley (1982, 1986a and 1986b) offered a systematic mathematical interpretation of the circuit of capital through the appropriate introduction of stock and flow variables representing the different forms that capital assumes throughout its circulation and the careful examination of the linkages and time lags between them. This interpretation turned out a breakthrough that allowed not only for a deeper exploration of the circuit of capital itself, but also the assessment of several related matters. Hence, Loranger (1989) attempted to provide a Marxian explanation for inflation revealing the influence of the interest rate and the exchange rate over the price level; Kotz (1991) and dos Santos (2011) considered credit trying to show its relevance in the process of accumulation, while dos Santos (2014) attempted to link this model to the distribution of income; Abeles (2012) extended the model from a closed to an open economy, discussing the implications of capital exports and imports. Foley and Alemi (1997), Matthews (2000), Alemi and Foley (2010), Shaikh (2012) and Basu (2014) offered econometric examples based on Foley’s interpretation, proving the strength of the model into explaining the rise and failure of productivity in different sectors of the economy, or the slowdown of the entire economy as it approached the 2008 crisis.

In what follows, the original model is reviewed, before a new formulation is proposed. Following dos Santos, we will also include the possibility of borrowing, though only for the case of financing investment and not consumption. Although we shall stick to the presentation by Foley (1986b), the notation will be slightly changed to make it compatible with Marx’s notation in the above presentation.

3.1. Foley's interpretation of the circuit of capital

First of all, it is important to distinguish between stock and flow variables; the productive capital, the commercial capital and the money capital described above are treated as stocks, hence as capital in static form, while we shall state a number of flow variables to treat the transformations from one form to the other.⁵ The first flow variables are the flow of value of capital outlays at time t , $O(t)$, and the flow of value of finished commodities at cost at time t , $Q(t)$; since the finished commodities are brought about by means of the capital outlays through the process of production, the two variables are related as

$$Q(t) = \int_{-\infty}^t O(t') \alpha_P(t - t'; t') dt' \quad (1)$$

where $\alpha_P(t - t'; t')$ is the time of production lags, or alternatively the proportion of the costs incurred at time t' which emerge as finished outputs at time t . Assuming a rate of return on costs, $\rho(t)$,⁶ we also introduce the flow of value of finished commodities at market at time t , $N(t)$, as

$$N(t) = \int_{-\infty}^t O(t') (1 + \rho(t')) \alpha_P(t - t'; t') dt' \quad (2)$$

Following, we introduce the flow of value of the realization (or total revenue) at time t , $R(t)$, that is the flow of value of the traded commodities. Since, realization follows from the value of the finished commodities, we have

$$R(t) = \int_{-\infty}^t N(t') \beta_R(t - t'; t') dt' \quad (3)$$

⁵ So as to distinguish between stock and flow variables, the former express capital in a static form, have units of value (either labor hours or money) and are denoted as above with non-italic symbols (P, C and M); the latter express a capital in motion, have units of value over time (*e.g.* labor hours over one year) and are denoted with italic symbols (Q , N , R and O).

⁶ Foley terms this a mark-up. However, in Classical and Marxian theory, this is more than a mere mark-up in the cost price, but an endogenous variable determined through the technical and social conditions of production and the intra-sectoral competition of capital – rather than just chosen by a particular capitalist as the mark-up could be. We avoid calling it the ‘rate of profit’, as this should be calculated over the entire advanced capital stock, not just the flow of capital outlays; the name ‘rate of revenue on costs’ is preferred as it mirrors the Sraffian rate of profit, defined only with respect to the circulating capital. We should note that this variable contains the Marxian ‘rate of surplus value’, hence it contains the measure of exploitation of the laborers by capitalists.

where $\beta_R(t - t'; t')$ the time profile of realizations lags, or alternatively the proportion of commodities finished at time t' which are sold at time t . From this flow of value, some part corresponds to the recovery of the expended capital outlays, including the depreciation of long-lived equipment; this is denoted $R'(t)$ and it is related to the cost of the finished commodities, thus it is given as

$$R'(t) = \int_{-\infty}^t Q(t')\beta_R(t - t'; t')dt' \quad (4)$$

The portion of the realization flow that remains forms the realized surplus value, which constitutes the difference of the value at market from the value at cost, thus it is given as

$$R''(t) = \int_{-\infty}^t (N(t') - Q(t'))\beta_R(t - t'; t')dt' \quad (5)$$

It should be obvious that $R(t) = R'(t) + R''(t)$; given that $N(t) \sim (1 + \rho(t))Q(t)$ for slow variations of $\rho(t)$, then it is crucial to see that

$$R'(t) \sim \frac{R(t)}{1 + \rho(t)} \quad \text{and} \quad R''(t) \sim \frac{\rho(t)R(t)}{1 + \rho(t)} \quad (6)^7$$

Finally, the flow of value of capital outlays should be given by means of the recommittal of money to production; these money may come from the realized value that corresponds to the recovery of expended capital outlays ($R'(t)$) and from the portion of realized surplus value ($R''(t)$) that is capitalized (invested). If we assume a propensity to invest, $s(t)$, then

$$O(t) = \int_{-\infty}^t (R'(t') + s(t)R''(t'))\gamma_F(t - t'; t')dt' \quad (7)$$

where $\gamma_F(t - t'; t')$ the time profile of financial lags, that is of lags in the recommittal of money to production, or alternatively the proportion of money exchanged for commodities at time t' which are reinvested at time t . For dos Santos (2011), the flow of value of capital outlays can be augmented by means of borrowing; however, we shall avoid this complication in this work.

⁷ These relations can be precise if and only if, either the definitions of equations (1)-(5) are simplified, or the rate of return on costs is assumed constant. However, the equations for $R'(t)$ and $R''(t)$ may be regarded as true at all cases, as they do form the two constituent components of the revenue.

We should note that the time profiles are positive real-valued functions, defined for $t > t'$, that denote a certain amount of value tied up in each phase of the circuit at any time t , such that

$$\int_{-\infty}^{+\infty} \alpha_P(t - t'; t') dt' = \int_{-\infty}^{+\infty} \beta_R(t - t'; t') dt' = \int_{-\infty}^{+\infty} \gamma_F(t - t'; t') dt' = 1 \quad (8)$$

and

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^t \alpha_P(t - t'; t') dt' &= \frac{1}{\tau_P} \\ \frac{d}{dt} \int_{-\infty}^t \beta_C(t - t'; t') dt' &= \frac{1}{\tau_R} \\ \frac{d}{dt} \int_{-\infty}^t \gamma_F(t - t'; t') dt' &= \frac{1}{\tau_F} \end{aligned} \quad (9)$$

where τ_P is the average production lag (related to labor productivity), τ_R is the average realization lag (related to the tightness of the commodity market) and τ_F is the average financial lag (related to the tightness of the money market) at time t – all of which are positive and greater than one.

Turning now to the stock variables, the change in the stock of the aggregate productive capital, $P(t)$, is given as the difference between the flows of value of capital outlays and finished commodities at cost, hence as

$$\dot{P}(t) = O(t) - Q(t) \quad (10)$$

Similarly, the change in the stock of the aggregate commercial capital, $C(t)$, is given as the difference between the flows of value of finished commodities at market and realization, hence as

$$\dot{C}(t) = N(t) - R(t)$$

However, given a slowly changing (or stationary) rate of return on costs, this can be expressed with respect to the flow of values of finished commodities at cost minus the recovery of the capital outlays, hence as

$$\dot{C}(t) = N(t) - R(t) \sim (1 + \rho(t))(Q(t) - R'(t)) \quad (11)$$

Finally, the change in the stocks of the aggregate money capital, $M(t)$, is given as the difference between the portion of realization directed to recovery of capital outlays an investment and flow of value of capital outlays, as

$$\dot{M}(t) = R'(t) + s(t)R''(t) - O(t) \quad (12)$$

Equations (10), (11) and (12) should be forming a closed system describing the entire circuit of capital; however, these equations are not closed as they require knowledge over the flow variables, two unknown variables (the rate of profit and the propensity to save) and a fourth stock variable. However, these three forms of capital are the faces with which it appears; thus the total social capital, $K(t)$, at any moment t must be the sum of all money that constitute money capital, all commodities that constitute commercial capital and all means of production and labor power that constitute productive capital – or

$$K(t) = M(t) + C(t) + P(t) \quad (13)$$

Thus, the rate of profit is defined as

$$r(t) = \frac{R''(t)}{K(t)} = \frac{R''(t)}{M(t) + C(t) + P(t)} \quad (14)$$

which, after some simplifying assumptions (Foley 1986a and 1986b) may be written as

$$r(t) = \frac{\rho(t)}{\tau_F + \tau_R + \tau_P}$$

that is, as the ratio of the rate of return on costs (which contains the rate of surplus value) over the turnover time of total capital stock (which is merely the sum of time lags in each phase of the circuit of capital).

Furthermore, adding equations (10), (11) and (12), we have

$$\dot{K}(t) = \dot{M}(t) + \dot{C}(t) + \dot{P}(t) = s(t)R''(t) + \rho(t)(Q(t) - R'(t))$$

which is the change in the total capital stock of the economy. The change is decomposed to

1. the investment out of realized surplus value, $s(t)R''(t)$, which is the driving force of growth, and
2. the change in stocks due to the difference between the flow of value of capital outlays and the flow of value for their recovery, $\rho(t)(Q(t) - R'(t))$; this change in stocks is equal to a portion of the change in the commercial capital, namely $\frac{\rho(t)}{1+\rho(t)}\dot{C}(t)$, as it is there where the excess or lack of stocks is observed.

In general, as depicted in the Cambridge equation, the growth rate –minus any devaluation of the capital stock– is proportional to the rate of profit;⁸ furthermore, as mentioned in Section 2, the commercial capital is the only form of capital whose circuit begins with the presently created surplus value integrated. In this light, we shall attempt rewrite the above equation for the total

$$\dot{M}(t) + (1 + \rho(t))^{-1}\dot{C}(t) + \dot{P}(t) = s(t)R''(t) \quad (14)$$

3.2. A new formulation

Our first concern is to define new dimensionless variables referring to the money capital, the ‘cleansed’ commercial capital and the productive capital, by using the constraint supplied by equation (14). We start by dividing the left-hand side of equation (14) with $s(t)R''(t)$ and define

$$\begin{aligned} x &= \frac{\dot{P}(t)}{s(t)R''(t)} \\ y &= \frac{(1 + \rho(t))^{-1}\dot{C}(t)}{s(t)R''(t)} \\ z &= \frac{\dot{M}(t)}{s(t)R''(t)} \end{aligned} \quad (15)$$

that express the change of each form of capital relative to the change in the total capital stock. Differentiating these variables with respect to time and using equations (10), (11) and (12), we have

$$\begin{aligned} \dot{x} &= x \left(\frac{\ddot{P}(t)}{\dot{P}(t)} - \frac{\dot{R}''(t)}{R''(t)} - \frac{\dot{s}(t)}{s(t)} \right) = \frac{\dot{O}(t) - \dot{Q}(t)}{s(t)R''(t)} - x \left(\frac{\dot{R}''(t)}{R''(t)} + \frac{\dot{s}(t)}{s(t)} \right) \\ \dot{y} &= y \left(\frac{\ddot{C}(t)}{\dot{C}(t)} - \frac{\dot{\rho}(t)}{1 + \rho(t)} - \frac{\dot{R}''(t)}{R''(t)} - \frac{\dot{s}(t)}{s(t)} \right) = \frac{\dot{Q}(t) - \dot{R}'(t)}{s(t)R''(t)} - y \left(\frac{\dot{\rho}(t)}{1 + \rho(t)} + \frac{\dot{R}''(t)}{R''(t)} + \frac{\dot{s}(t)}{s(t)} \right) \end{aligned}$$

⁸ In its standard form ($g = sr$), the Cambridge equation does not contain the rate of devaluation; however, we have argued elsewhere that the picture of capital accumulation is incomplete without this addition (Chatzarakis *et al.* 2022). We should note that devaluation is significantly different from depreciation. The latter is a purely accounting concept, dealing with the financial amortization of stocks; the former though is a concept introduced by Marx (*Capital* III, ch. 6), related with the conditions of the market. What we argue here is that the change in stocks coming from ‘leftovers’ of the commercial capital constitutes in fact a devaluating force to the rate of growth – and thus we can ‘cleanse’ the changes of the capital stock by merely subtracting this term from it.

$$\dot{z} = z \left(\frac{\ddot{M}(t)}{\dot{M}(t)} - \frac{\ddot{R}''(t)}{R''(t)} - \frac{\dot{s}(t)}{s(t)} \right) = \frac{\dot{R}'(t) - \dot{O}(t)}{s(t)R''(t)} - (z - 1) \left(\frac{\dot{R}''(t)}{R''(t)} + \frac{\dot{s}(t)}{s(t)} \right)$$

At this point, the system can be written in a closed autonomous form, so long as we can allow for some simplifications.

First of all, let us examine the following three fractions

$$\frac{\dot{O}(t)}{s(t)R''(t)}, \quad \frac{\dot{Q}(t)}{s(t)R''(t)} \quad \text{and} \quad \frac{\dot{R}'(t)}{s(t)R''(t)}$$

Given equations (6), (7) and (9), the first fraction becomes

$$\frac{\dot{O}(t)}{s(t)R''(t)} = \frac{1}{\tau_F} \left(\frac{R'(t)}{s(t)R''(t)} + 1 \right) = \frac{1}{\tau_F} \left(\frac{1}{s(t)\rho(t)} + 1 \right)$$

which means that the change of the flow of value of capital outlays over the flow of value of capitalized surplus value is related to the financialization lag, the propensity to save and the rate of return on costs, or equivalently to the financialization lag and the rate of expansion on costs, $s(t)\rho(t)$; essentially, the greater the expansion rate is and thus the faster the accumulation of capital, the slower is the exhaustion of the capital outlays. Indeed, accumulation of capital stock raises the productive capacity and the labor productivity, leading to a more rational (labor-saving and material-saving) use of resources and a slower devaluation of the equipment. As discussed in Section 1, this process is related to the amount of productive capital, as well as to the state of the financial markets; subsequently, we may express the first fraction as a function of x and z , as

$$\frac{\dot{O}(t)}{s(t)R''(t)} = \frac{1}{\tau_F} \left(\frac{1}{s(t)\rho(t)} + 1 \right) = \frac{1}{\tau_F} (f_1(x, z) + 1) \quad (16)$$

such that at least one stationary point (preferably maximum) exists – hence such that

$$\left(\frac{\partial f_1}{\partial x} \right)_{\{x^*, z^*\}} = \left(\frac{\partial f_1}{\partial z} \right)_{\{x^*, z^*\}} = 0$$

for some x^* and z^* , that can be perceived as the point of ‘normal’ accumulation; and such that the expansion is slow and smooth over the accumulation of capital, *i.e.* that it is always concave – or

$$\frac{\partial^2 f_1}{\partial x^2} \leq 0 \quad \text{and} \quad \frac{\partial^2 f_1}{\partial z^2} \leq 0$$

which generally means that

$$\begin{cases} \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial z} > 0 & \text{near } x = z = 0 \\ \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial z} < 0 & \text{near } x = 1 \text{ or } z = 1 \end{cases}$$

Note that the $f_1(x, z) > 0$ for $x, z > 0$, unless the rate of expansion on costs explodes.⁹

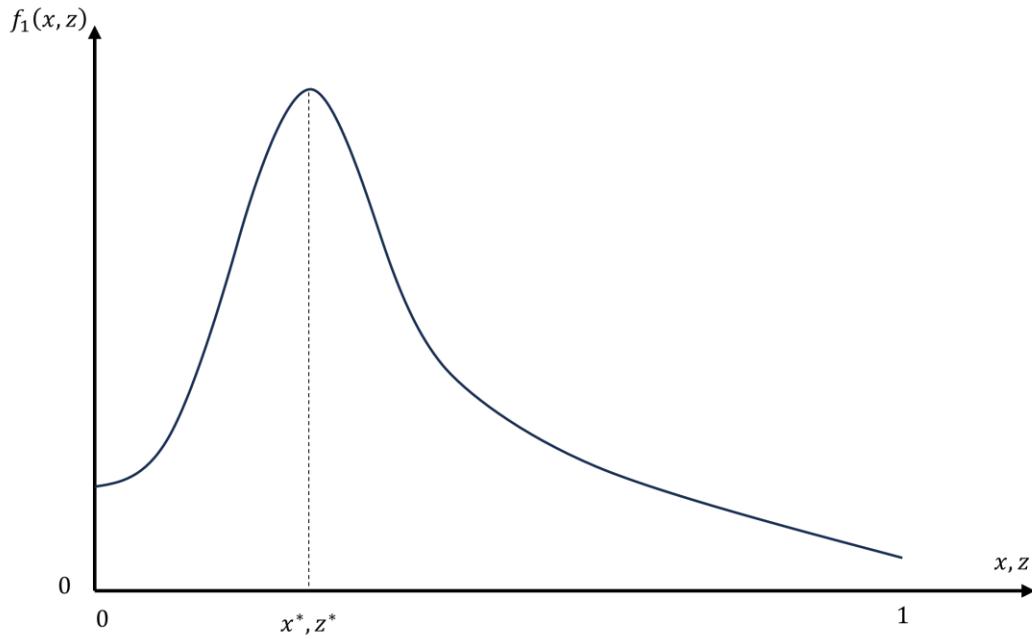


Figure 1: The shape of the ‘financing function’ $f_1(x, z)$, where the two independent variables are compressed in one axis.

Given equations (1) and (6), the second fraction becomes

⁹ Functions such as this can be found in the literature of ecological and epidemiological models, describing phenomena known as saturation (see, *e.g.* Tang *et al.* 2008). For example,

$$f_1(x, z) = A \frac{y^k}{1 + \left(\frac{y}{y^*}\right)^l} e^{B(x-x^*)^m}$$

for which, (x^*, y^*) is a maximum for $k < l$, and a stationary point for $k > l$; note that all parameters are positive.

$$\frac{\dot{Q}(t)}{s(t)R''(t)} = \frac{1}{\tau_p} \frac{O(t)}{s(t)R''(t)}$$

hence the change of the flow of value of finished commodities at cost is related to the production lag and the flow of value of capital outlays; essentially, the use of capital outlays over the average production lags is translated to a change in the flow of finished commodities – or, the more intense and more rapid the process of production is, the more finished commodities are produced. The use of capital outlays during the process of production depends on the technical conditions of production, hence the level and the changes of the stock of productive capital. To capture this, we shall assume that the change in the flow of value of finished commodities at cost is a function of x , as

$$\frac{\dot{Q}(t)}{s(t)R''(t)} = \frac{1}{\tau_p} f_2(x) \quad (16)$$

such that

$$\frac{\partial f_2}{\partial x} \geq 0 \quad \text{and} \quad \frac{\partial^2 f_2}{\partial x^2} \leq 0$$

Note that, so long as the process of production does not stop, the flow of capital outlays must always be positive, even if disinvestment occurs, hence $f_2(x) > 0$ for $x > 0$.¹⁰

¹⁰ Although the notion and the usage of a production function is heavily criticized and rightfully so (Shaikh 1974), this function could be a classical analogue of it; a better analogue though would be Kaldor's (1957) 'technical progress function', which related the growth of output (and thus the change of $Q(t)$) to the growth of capital stock (and thus to x); this function is also positive, increasing and concave.

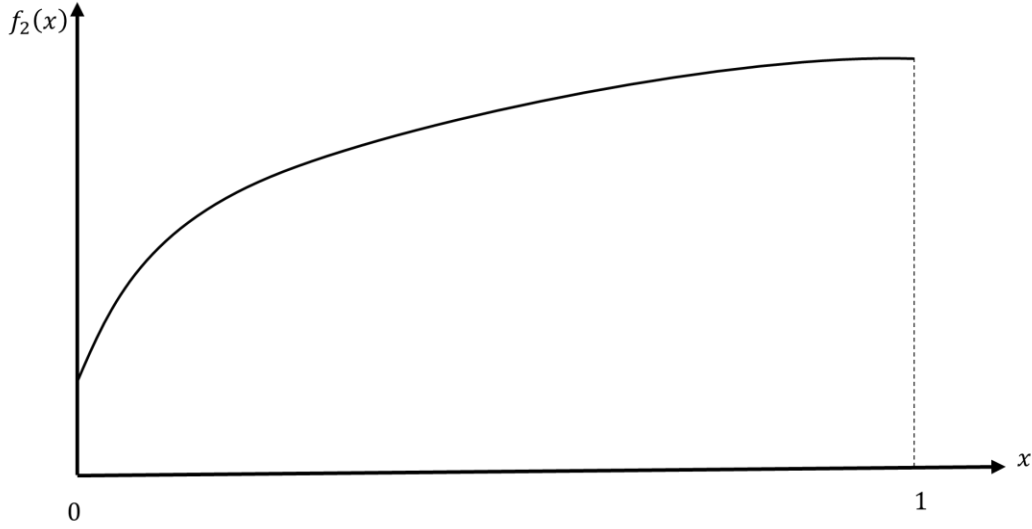


Figure 2: A typical representation of the ‘production function’ $f_2(x)$.

Given equations (4), (6) and (7), the third fraction is written as

$$\frac{\dot{R}'(t)}{s(t)R''(t)} = \frac{1}{\tau_R} \frac{Q(t)}{s(t)R''(t)} = \frac{1}{\tau_R} \left(\frac{\dot{C}(t)}{(1 + \rho(t))s(t)R''(t)} + \frac{1}{s(t)\rho(t)} \right)$$

that is, the change in the flow of value of realization corresponding to the recovery of capital outlays is expressed as the flow of value of capital outlays over the realization lag – or, equivalently, directly related to y and to the rate of expansion on costs, $s(t)\rho(t)$. Indeed, the change in the flow of value that recovers the capital outlays is positively related to the conditions of trade and inversely related to the growth rate. Thus,

$$\frac{\dot{R}'(t)}{s(t)R''(t)} = \frac{1}{\tau_R} (y + f_1(x, z)) \quad (16)$$

Let us now proceed to the terms

$$\frac{\dot{\rho}(t)}{1 + \rho(t)} \quad \text{and} \quad \frac{\dot{R}''(t)}{R''(t)} + \frac{\dot{s}(t)}{s(t)}$$

As for the first term, it describes the dynamics of the rate of return on costs. This variable is determined by the social and technical conditions of production and contains the rate of surplus

value, hence the measure of exploitation of laborers by capitalists; in general, it represents the ‘productivity’ of the circulating capital used in the process of production, or equivalently the pace by which the human and material inputs of the process of production yield revenue. Thus, it rises with the expansion of the productive capital, as the productive capacity and the labor productivity themselves rise. To simplify the expression, we notice that for $\rho(t) < 1$

$$\frac{1}{1 + \rho(t)} \cong 1 - \rho(t) + \rho^2(t) - \rho^3(t) + \rho^4(t) - \rho^5(t) + \dots$$

hence

$$\frac{\dot{\rho}(t)}{1 + \rho(t)} \cong \dot{\rho}(t)(1 - \rho(t) + \rho^2(t) - \rho^3(t) + \rho^4(t) - \rho^5(t) + \dots)$$

and, generally, the first terms may be a sufficient approximation, as the rate of return on costs is generally below well 0.5 (for example, the corporate profit margins in the US are between 5% and 10% and in Japan between 2% and 6%). As noted, this is positively related to the amount of productive capital, hence we could simply have

$$\frac{\dot{\rho}(t)}{1 + \rho(t)} \cong \dot{\rho}(t)(1 - \rho(t)) = af_3(x) \quad (17)$$

where a some positive real-valued constant¹¹ and $f_3(x)$ a real-valued positive function such that $f_3(0) = f_3(1) = 0$ and

$$\frac{\partial^2 f_3}{\partial x^2} \leq 0$$

to imitate the fact that the rate of return on costs –as well as the rate of surplus value– would always rise with an acceleration of accumulation, but would remain constant in the extreme cases of a stationarity or domination of the productive capital; furthermore, due to its concave form, we can safely assume that its maximum is at the point of ‘normal’ accumulation – or

$$\left(\frac{\partial f_3}{\partial x}\right)_{\{x^*\}} = 0$$

¹¹ Obviously, the choice of $a = 0$ means that the rate of revenue on costs remains constant over time. Important to note is that this does not mean that the rate of profit does not change over time.

Moving on to the second term, this describes the dynamics of investment, decomposed to the dynamics of realized surplus value and the dynamics of the propensity to save. The realization of surplus value is regulated by the several factors, including the realization time lag, the state of commercial capital, the rate of return on costs, *etc.*; indeed, using equations (5) and (11) we may write

$$\begin{aligned}\frac{\dot{R}''(t)}{R''(t)} &= \frac{1}{\tau_R} \frac{N(t) - Q(t)}{R''(t)} = \frac{1}{\tau_R} \frac{\rho(t)Q(t)}{R''(t)} = \frac{1}{\tau_R} s(t)\rho(t) \frac{Q(t)}{s(t)R''(t)} = \\ &= \frac{1}{\tau_R} \left(s(t)\rho(t) \frac{\dot{C}(t)}{(1 + \rho(t))s(t)R''(t)} + 1 \right)\end{aligned}$$

hence, using the definitions given,

$$\frac{\dot{R}''(t)}{R''(t)} = \frac{1}{\tau_R} \left(\frac{y}{f_1(x, z)} + 1 \right)$$

The propensity to save expresses the so-called ‘animal spirits’ of capitalists, who are willing to invest when the conditions of the market are thriving and hesitate when the markets become tight. According to Harrod’s ‘knife-edge’ problem, these ‘animal spirits’ are positively related to the productive capacity of the economy; however, this productive capacity is viewed from the eyes of the capitalists, hence it does not refer to the accumulation of productive capital *per se*, but rather to the accumulation of money capital in the prospect of purchasing means of production and labor power. To capture this, we assume that the rate of change of the propensity to save depends on the financial time lag, representing the (in)ability of the financial markets to proceed with investments, and to a function of z , representing capitalists’ investment decisions based on the accumulation of money capital. Thus,

$$\frac{\dot{s}(t)}{s(t)} = \frac{1}{\tau_F} f_4(z)$$

Obviously, when the fraction of money capital is stationary, the propensity to save must be constant, hence $f_4(0) = 0$, while when the money capital dominates, the propensity to save must be maximized, let it be $f_4(1) = 1$. Furthermore, one should notice that investment decisions based on the capacity of capital often lead to opposite and undesired results, hence the function $f_4(z)$

should change its behavior about the point of ‘normal’ accumulation; assuming the function to be increasing as the accumulation of money capital accelerates, hence

$$\frac{\partial f_4}{\partial z} \geq 0$$

we can assume it has a stationary inflection point there, hence

$$\left(\frac{\partial f_4}{\partial z}\right)_{\{z^*\}} = \left(\frac{\partial^2 f_4}{\partial z^2}\right)_{\{z^*\}} = 0$$

Moreover, we can admit that the endpoints are minimum and maximum respectively, as escaping either stagnation is impossible without some exogenous drive, hence

$$\left(\frac{\partial f_4}{\partial z}\right)_{\{0\}} = \left(\frac{\partial f_4}{\partial z}\right)_{\{1\}} = 0^{12}$$

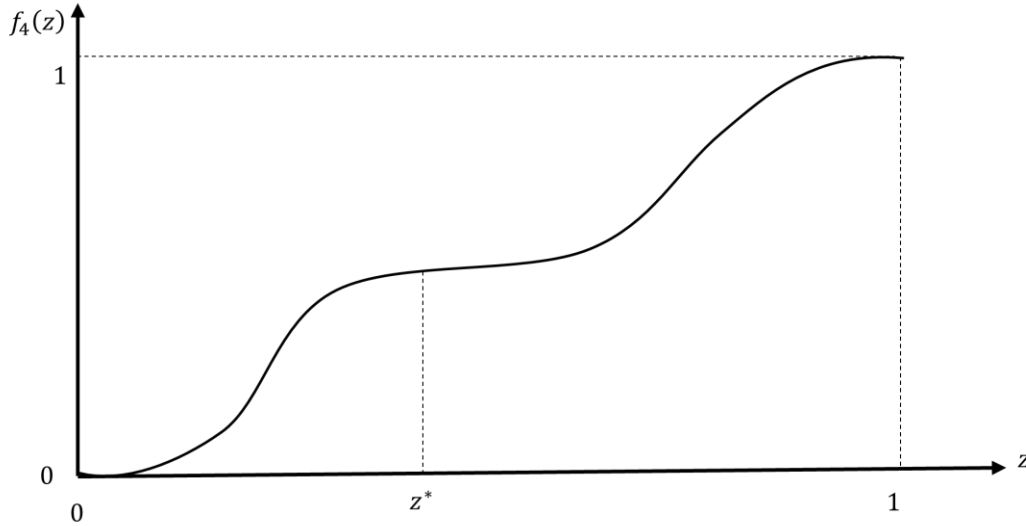


Figure 3: The proposed form of the ‘investment function’ $f_4(x)$.

¹² This function is similar to two joined logistic curves; the first rising slowly for $z \rightarrow 0$ and rapidly as $z \rightarrow z^*$, and the second rising slowly for $z \rightarrow z^*$ and rapidly as $z \rightarrow 1$. This could imitate capitalists’ ‘rational’ aversion *à la* Harrod of the ‘normal accumulation’ point, as well as the difficulty of

As a result, the term can be expressed as

$$\frac{\dot{R}''(t)}{R''(t)} + \frac{\dot{s}(t)}{s(t)} = \frac{1}{\tau_R} \left(\frac{y}{f_1(x, z)} + 1 - \frac{\tau_R}{\tau_F} f_4(z) \right) \quad (18)$$

In conclusion, the ordinary differential equations governing the evolution of x , y and z are

$$\dot{x} = \frac{1}{\tau_F} f_1(x, z) + \frac{1}{\tau_F} - \frac{1}{\tau_P} f_2(x) - \frac{1}{\tau_R} x \left(\frac{y}{f_1(x, z)} + 1 - \frac{\tau_R}{\tau_F} f_4(z) \right) \quad (19)$$

$$\dot{y} = \frac{1}{\tau_P} f_2(x) - \frac{1}{\tau_R} f_1(x, z) - \frac{1}{\tau_R} y - \frac{1}{\tau_R} y \left(\tau_R a f_3(x) + \frac{y}{f_1(x, z)} + 1 - \frac{\tau_R}{\tau_F} f_4(z) \right) \quad (20)$$

$$\dot{z} = \frac{1}{\tau_R} y + \left(\frac{1}{\tau_R} - \frac{1}{\tau_F} \right) f_1(x, z) - \frac{1}{\tau_F} - \frac{1}{\tau_R} (z - 1) \left(\frac{y}{f_1(x, z)} + 1 - \frac{\tau_R}{\tau_F} f_4(z) \right) \quad (21)$$

with a constraint supplied by equation (14), as

$$x + y + z = 1 \quad (22)$$

We should note that, summing the three differential equations,

$$\dot{x} + \dot{y} + \dot{z} = -a y f_3(x)$$

whereas the logical result would be 0; this would be mendable if $a = 0$, or equivalently that the rate of return on costs is constant (or at least stationary) over time. This is an assumption made in Foley (1986a and 1986b), however one that seems meaningful in the long run, as we shall see.

4. Analyzing the model

Notably, if $a = 0$ (a reasonable assumption in the short run), the system of equations (19)-(22) is an adjusted version of the well-known SIRS epidemiological model. These models describe the process of a disease spreading in the population of a closed country. The population is affected in a cyclical manner, such that the initial population are all unaffected, but susceptible (S) to the disease; once exposed to it, some of them become infected (I) and after some time either recover (R) or die;¹³ the recovered, depending on whether the disease allows for antibodies, may become

¹³ Some refer to the third variable as removed, so that both recovered and deceased fit into it. In this case, the three variables always add to one.

susceptible again and thus reopen a circle of infection. These models are well studied in the relevant literature –we may refer *inter alia* to Liu *et al.* (1986), Liu *et al.* (1987), Sharp and Pastor (2011) for results on these models– and are known to have a specific structure. In most (simple) cases, the system is proved to have a trivial equilibrium point (where the susceptible constitute the entire population and the disease disappears), which is unstable, and a non-trivial equilibrium point, which is (locally or globally) asymptotically stable. In non-linear cases, the system may give rise to one or more limit cycles and the solutions may fluctuate indefinitely around the non-trivial equilibrium. In our case, we could treat z as the susceptible, x as the infected, and y as the recovered (or removed); indeed, the circuit of capital initiates with the money capital, continues with the productive capital and is completed with the commercial capital.

4.1. The equilibrium points

To analyze the model of equations (19)-(22), we will start by locating the equilibrium points and addressing their stability. Setting $\dot{x} = \dot{y} = \dot{z} = 0$, we notice that there are (at least) two equilibrium points. First, we will seek for trivial ones, where two of the variables are zero and the remaining equal to one.

The first of them expresses an abundance and rapid accumulation of productive capital, which we understand as an ‘excess capital’ or ‘overaccumulation’ stagnation point:

$$OA: \{x^* = 1, y^* = z^* = 0\} \quad (23)$$

This point is indeed an equilibrium so long as the assumed $f_1(1,0) = \frac{2\tau_R - \tau_F}{\tau_F - \tau_R}$, and $f_2(1) = \frac{\tau_P(2\tau_R - \tau_F)}{\tau_R(\tau_F - \tau_R)}$ asymptotic states hold, which are positive –and hence consistent– if-f $\tau_F > \tau_R$. This point is unstable (more specifically, a saddle), since two of the eigenvalues have positive real parts.¹⁴ This inequality may safely be assumed to hold at all times, since the left-hand side is most likely positive and the right-hand side is most likely negative; thus, the ‘overproduction’ equilibrium point can be safely assumed unstable for all cases.

¹⁴ These results were obtained utilizing the Routh-Hurwitz criterion. For a short discussion on the calculations, see Appendix A.

Similarly, if the $f_1(0,0) = \frac{\tau_F}{\tau_R - \tau_F}$ and $f_2(0) = \frac{\tau_P \tau_R}{\tau_F(\tau_R - \tau_F)}$ asymptotic states hold, then a second trivial equilibrium exists, expressing an abundance and rapid accumulation of commercial capital; this one we understand as an ‘excess commodities’ or ‘underconsumption’ stagnation point:

$$\text{EC: } \{x^* = 0, y^* = 1, z^* = 0\} \quad (24)$$

There asymptotic states are positive –and hence consistent– if-f $\tau_R > \tau_F$. Once again, this equilibrium point is becomes unstable (again, a saddle) when

$$\left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}} > 3 + \frac{\tau_F}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}} + \left(\frac{\tau_R}{\tau_F} + 1\right) \left(\frac{\partial f_1}{\partial z}\right)_{\{0,0\}} > 0$$

as two of the eigenvalues have positive real parts. Assuming that the derivatives of $f_1(x, z)$ are of the same magnitude, the ‘excess commodities’ point can hardly be unstable; so it is generally assumed stable.

The case of an ‘excess money’ or ‘financialization’ stagnation, where an abundance and rapid accumulation of money capital coincides with stationarity of productive and commercial capitals, is also possible

$$\text{EM: } \{x^* = y^* = 0, z^* = 1\} \quad (25)$$

if the $f_1(0,1) = \frac{\tau_R}{\tau_F - \tau_R}$ and $f_2(0) = \frac{\tau_P}{\tau_R - \tau_F}$ asymptotic states hold, which are positive –and hence consistent– if-f $\tau_F > \tau_R$. This equilibrium point turns unstable (a saddle) when

$$\frac{\tau_P}{2\tau_F} \left(\frac{3\tau_F - 2\tau_R}{\tau_R} + \left(\frac{\partial f_1}{\partial x}\right)_{\{0,1\}} + 2 \frac{\tau_F - \tau_R}{\tau_R} \left(\frac{\partial f_1}{\partial z}\right)_{\{0,1\}} \right) > \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}}$$

as two of the eigenvalues have positive real parts; however, the left-hand side is negative, while the right-hand side is positive. So, the ‘excess money’ point must be stable.

Finally, at least another, non-trivial equilibrium point exists, that expresses the conditions for normal reproduction of the capitalist mode of production; however, having not specified the functions $f_1(x, z)$, $f_2(x)$, $f_3(x)$ and $f_4(z)$, we cannot determine this point analytically. What we can determine, however, is that at this point, the productive capacity of capital must be maximum

and the capitalists' 'animal spirits' are not yet effective; hence, all $f_1(x, z)$, $f_3(x)$ and $f_4(z)$ are at their stationary point. As a result, the equilibrium point of normal reproduction,

$$\text{NR: } \{x^*, y^*, z^*\}$$

marks a case where productive, commercial and money capital are at a balance; note that none of them is zero and $x^* + y^* + z^* = 1$. This is the case Foley (1986a and 1986b) discussed under some simplifications of his original model.

An interesting case is if $y^* = f_1(x^*, z^*)$, whereas the change of commercial capital is equal to the total surplus value. In other words, the circulation of commodities expands equally to the availability of surplus value, hence (in Keynesian wording) the final demand that causes economic expansion is capitalists' demand for consumption and investment; this captures the very essence of Marx's schemes of expanded reproduction (Tsoulfidis and Tsaliki 2019, ch. 2), deeming this condition an essential for this equilibrium point.

Another notable condition that appears for this equilibrium, if- $y^* = f_1(x^*, z^*)$, is that $f_3(x^*) = 0$, hence the rate of return on revenue (and thus the rate of surplus value) is constant.¹⁵ Henceforth, we shall consider only this case.

The asymptotic analysis reveals that this equilibrium is always attracting since

$$\frac{2}{\tau_R} + \frac{1}{\tau_F} f_4(z^*) \quad \text{and} \quad \frac{3}{\tau_R} + \frac{1}{2\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} + \frac{1}{\tau_F} f_4(z^*)$$

are always positive; it can also shift between monotonic and oscillatory behavior depending on whether

$$4 \frac{\tau_P}{\tau_R} - \frac{4(x^* + y^*)}{y^*} \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} + \frac{\tau_R}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}}^2$$

is positive or negative – the latter happening mainly for sufficiently large x^* (and thus sufficiently small y^*) and being the dominant case if $\tau_P < \tau_R$. Hence, the faster the process of production is

¹⁵ If our assumptions for $f_3(x)$ hold and this is true, then both the minima ($f_3(0)$ and $f_3(1)$) and the maximum ($f_3(x^*)$) are identically zero. In this case, the function itself must be zero, deeming the rate of return on costs constant, exactly as Foley assumed; furthermore, now $\dot{x} + \dot{y} + \dot{z} = 0$ as would be expected.

with respect to commerce ($\tau_P < \tau_R$) and the larger the accumulation rate of productive capital ($x^* > y^*$), the more likely the NR equilibrium point to be a stable focus.

4.2. The limit cycle

As noted, the system has a constraint, $x + y + z = 1$; a constraint that all its equilibrium points fulfil. Thus, the solutions of the system should not be spread over entire 3-dimensional phase space, but be restricted on this 2-dimensional plane. This can be used to reduce the system from a 3- to a 2-dimensional one, centered on the NR equilibrium, through the linear transformation

$$u = (1 - x) - (1 - x^*) = x^* - x \quad \text{and} \quad v = 1 - \frac{1}{2}(x - x^* + y - y^*) = \frac{1}{2}(3 + z - z^*)$$

where $\dot{u} = -\dot{x}$ and $\dot{v} = \frac{1}{2}\dot{z}$.¹⁶ Denoting $F_1(u, v)$, $F_2(u)$ and $F_4(v)$ the functions $f_1(x - x^*, z - z^*)$, $\frac{\tau_R}{\tau_P}f_2(x - x^*)$ and $\frac{\tau_R}{\tau_F}f_4(z - z^*)$ expressed with respect to the new variables, the system becomes

$$\dot{u} = -\frac{1}{\tau_F} - \frac{1}{\tau_F}F_1(u, v) - \frac{1}{\tau_R}F_2(u) - \frac{1}{\tau_R}(\alpha - u) \left(1 + \frac{\beta + u - 2v}{F_1(u, v)} - F_4(v) \right) \quad (26)$$

$$\begin{aligned} \dot{v} = & -\frac{1}{2\tau_F} + \frac{1}{2\tau_R}(u - 2v) + \frac{1}{2} \left(\frac{1}{\tau_R} - \frac{1}{\tau_F} \right) F_1(u, v) \\ & - \frac{1}{2\tau_R}(2v - \alpha - 2\beta) \left(1 + \frac{\beta + u - 2v}{F_1(u, v)} - F_4(v) \right) \end{aligned} \quad (27)$$

where $\alpha = 2 - x^*$ and $\beta = x^* + z^* + 1$. Notably, τ_P can be explicitly removed from the equations, rendering the system independent to the time profile of the production process.¹⁷

The study of the ‘normal reproduction’ point, (u^*, v^*) , and of the global behavior of the system are more easily conducted in terms of equations (26) and (27). Starting with the former, the linearized system in the vicinity of NR is

¹⁶ Note that the new variables, u and v are not strictly positive, or restricted in $[0,1]$, but they are equal to 0 in the NR equilibrium, which is of greater interest to us.

¹⁷ This may initially appear strange, as the production process is the focus of Marx’s thought; it is in the production process where use and exchange values are materialized and it is there that the fundamental contradictions of the capitalist mode of production are rooted and eventually revealed. However, it should not surprise us, as this is a model for the sphere of circulation, where the production process appears merely as a disruption. Notably, the production process can be merely seen as a ‘momentary’ disruption of the circulation process.

$$\begin{bmatrix} \dot{\Delta u} \\ \dot{\Delta v} \end{bmatrix} = \begin{bmatrix} -\left(\frac{\partial F_2}{\partial u}\right)_{\{0\}} - \frac{1}{\tau_R} F_4(0) - \frac{1}{\tau_R} \left(2 - \frac{\alpha}{\beta}\right) & \frac{1}{\tau_R} \left(\frac{\partial F_4}{\partial v}\right)_{\{0\}} + \frac{2}{\tau_R} \frac{\alpha}{\beta} \\ \frac{1}{2\tau_R} (\alpha + 6\beta) & -\frac{1}{2\tau_R} (\alpha + 2\beta) \left(\frac{\partial F_4}{\partial v}\right)_{\{0\}} + \frac{1}{\tau_R} F_4(0) - \frac{1}{\tau_R} \left(\frac{\alpha}{\beta} + 11\right) \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

where Δu , Δv are small linear perturbations of the respective variables. According to the Ruth-Hurwitz criterion, the NR point is stable if-f the following conditions hold

$$\begin{aligned} & -\left(\frac{\partial F_2}{\partial u}\right)_{\{0\}} - \frac{1}{2\tau_R} (\alpha + 2\beta) \left(\frac{\partial F_4}{\partial v}\right)_{\{0\}} - \frac{1}{\tau_R} \left(2 \frac{\alpha}{\beta} + 9\right) < 0 \\ & 3 \frac{\alpha}{\beta} + 13F_4(0) + 2F_4^2(v^*) + \tau_R(\alpha + 11\beta - F_4(0)) \left(\frac{\partial F_2}{\partial u}\right)_{\{0\}} + \tau_R \left(\frac{\alpha}{2} + \beta\right) \left(\frac{\partial F_2}{\partial u}\right)_{\{0\}} \left(\frac{\partial F_4}{\partial v}\right)_{\{0\}} \\ & > 22 + F_4^2(0) + \left(3\alpha + 2\beta + \left(\frac{\alpha}{2} + \beta\right) F_4(0)\right) \left(\frac{\partial F_4}{\partial v}\right)_{\{0\}} \end{aligned}$$

Notable the first condition always holds, given that all parameters involved are positive; then, the two conditions can be combined in

$$\frac{2(1 - u^*)^2}{3(1 + u^* - 2v^*)^2} - \frac{20(1 - u^*)}{1 + u^* - 2v^*} - \frac{1}{3} > \tau_R \left(1 + \frac{4(1 - u^*)}{1 + u^* - 2v^*}\right) \left(\frac{\partial F_2}{\partial u}\right)_{\{u^*\}}$$

which (given that the right-hand side is generally positive) can hold only for those values of u^* and v^* that fulfil

$$\frac{2(1 - u^*)^2}{3(1 + u^* - 2v^*)^2} > \frac{20(1 - u^*)}{1 + u^* - 2v^*} + \frac{1}{3}$$

which can generally hold for all possible values of u^* (between 0 and 1) and v^* (between 0.5 and 1).¹⁸ So, the NR equilibrium point is stable. Note that the off-diagonal elements of the matrix are

¹⁸ In fact, the right-hand side can be as larger as 0 as possible. Solving the inequality

$$\frac{2(1 - u^*)^2}{3(1 + u^* - 2v^*)^2} - \frac{20(1 - u^*)}{1 + u^* - 2v^*} - \frac{1}{3} > \Psi$$

where Ψ an arbitrarily large positive number, gives an increasing quasi-linear relation between u^* and v^* , that allows all admissible values for them.

(generally) of opposite sign, thus its eigenvalues are complex and the solutions of the system near the NR point are oscillatory.

However, this system undergoes a bifurcation¹⁹ when

$$\left(\frac{\partial F_2}{\partial u}\right)_{\{u^*\}} + \frac{3}{\tau_R} F_4(v^*) = \frac{1}{\tau_R} \left(\frac{13}{2} + \frac{1 - u^*}{1 + u^* - 2v^*} \right)$$

since the trace of the matrix turns zero, thus its eigenvalues become purely imaginary; the latter means that around the equilibrium of the system a limit cycle may appear.²⁰ Now, using the fact that

$$\frac{13}{2} - \frac{1 - u^*}{1 + u^* - 2v^*} > 3F_4(v^*)$$

the derivative of $F_2(u)$ in the vicinity of the NR equilibrium is bounded as

$$\left(\frac{\partial F_2}{\partial u}\right)_{\{u^*\}} < \frac{2}{\tau_R} \frac{1 - u^*}{1 + u^* - 2v^*}$$

Considering this a generic upper bound that may be true to any $\frac{\partial F_2}{\partial u}$, and using Grönwall's inequality,²¹ the latter can be integrated to the following upper bound for $F_2(u)$

$$F_2(u) < F_2(0) + \frac{2}{\tau_R} \frac{1 - u^*}{1 + u^* - 2v^*} u$$

Transforming these constraints to the original variables, we have

$$f_4(z^*) < \frac{13}{6} - \frac{x^*}{3y^*} \quad (28)$$

¹⁹ This bifurcation appears as an Andronov-Hopf bifurcation, however the stability of the OA point does not change. This indicates that this is not simply the generation of a limit cycle and we have to look further into it.

²⁰ A limit cycle is a closed periodic orbit that separates the phase space (here, the $x + y + z = 1$ plane) in regions of different behavior, *i.e.*, in regions where the system is explosive (solutions diverge) and regions where the system is dissipative.

²¹ The Grönwall lemma is a fundamental estimate for (nonnegative) functions on one real variable satisfying a certain differential inequality. In a nutshell, consider the differential inequality, $\psi'(t) \leq V(t)\psi(t)$; then its solution is bounded from above by the estimate $\psi(t) \leq \psi(0)e^{\int_0^t V(s)ds}$. Thus, a simplified version of an ordinary differential equation can be solved to acquire an upper bound for the solution of the complete version. In the case of partial differential equations, a generalization was proposed by Snow (1972).

which is always true as $0 < f_4(z^*) < 1$ and $\frac{13}{6} - \frac{x^*}{3y^*} > 1$; and

$$f_2(x) < \frac{1}{\tau_R} \left(\frac{\tau_P(2\tau_R - \tau_F)}{\tau_F - \tau_R} + 2 \frac{x^*}{y^*} (1 - x) \right) \quad (29)$$

which is also always true as $f_2(x)$ is increasing over x until it reaches $f_1(1) = \frac{\tau_P(2\tau_R - \tau_F)}{\tau_R(\tau_F - \tau_R)}$. Hence, by the assumptions of the functions $f_2(x)$ and $f_4(z)$, the NR equilibrium point is a stable focus (attracting solutions of the system by means of oscillations) and it undergoes an Andronov-Hopf bifurcation, allowing for the creation of at least one limit cycle.

This seems contradictory, since, according to the Poincaré-Bendixson theorem, the equilibrium point of a planar dynamical system should change its stability to admit a limit cycle (*e.g.*, a stable focus should turn to an unstable focus, so that a stable limit cycle may emerge around it). Our suspicion is that for the stability of the point to be retained, two limit cycles (a small stable and a larger unstable) must be formed simultaneously. To clarify this, we use the Bendixson-Dulac criterion and compute the divergence of the system of equations (26) and (27), which is

$$\begin{aligned} \frac{\partial \dot{u}}{\partial u} + \frac{\partial \dot{v}}{\partial v} = & - \left(\frac{1}{\tau_F} + \frac{2}{\tau_R} \frac{(1-u)(1+u-2v)}{F_1^2(u,v)} \right) \frac{\partial F_1}{\partial u} \\ & - \left(\frac{1}{2\tau_R} - \frac{1}{2\tau_F} + \frac{2}{\tau_R} \frac{(1-v)(1+u-2v)}{F_1^2(u,v)} \right) \frac{\partial F_1}{\partial v} - \frac{1}{\tau_R} \frac{3+5u-8v}{F_1(u,v)} + \frac{\partial F_2}{\partial u} \\ & + \frac{1}{\tau_R} \left(3F_4(v) - (1-v) \frac{\partial F_4}{\partial v} \right) - \frac{5}{2\tau_R} \end{aligned}$$

The latter determines whether the behavior of the system and, in particular, whether the solutions of the system converge or diverge within a region of the phase space. Namely, if $\frac{\partial \dot{u}}{\partial u} + \frac{\partial \dot{v}}{\partial v} < 0$, solutions converge (the system is dissipative) and if $\frac{\partial \dot{u}}{\partial u} + \frac{\partial \dot{v}}{\partial v} > 0$, solutions diverge (the system is explosive); if, however, the divergence is zero on some curve, then this is an invariant curve of the system; moreover, if this is a closed curve, it is a limit cycle of the system (a closed periodic trajectory to which other solutions may converge).²²

²² Another way to deal with this problem is to use the methodology provided by Li and Muldowney (1993). For the application of this method, see Appendix B. Unfortunately, results are conclusive only so far as to confirm the possibility for limit cycles to exist; the complexity of the model does not allow any further specification.

Setting $\frac{\partial \dot{u}}{\partial u} + \frac{\partial \dot{v}}{\partial v} = 0$ and assuming known $F_2(u)$ and $F_4(v)$, this is a partial differential equation for $F_1(u, v)$; its solutions would be possible forms for $F_1(u, v)$, for which the system of equations (26) and (27) has at least one invariant curve. Solving it is not particularly easy in the generic case. However, using Grönwall's inequality and the upper bounds for $F_4(v)$ and $\frac{\partial F_2}{\partial u}$, instead of solving

$$\begin{aligned} & \left(\frac{1}{\tau_F} + \frac{2}{\tau_R} \frac{(1-u)(1+u-2v)}{F_1^2(u, v)} \right) \frac{\partial F_1}{\partial u} + \left(\frac{1}{2\tau_R} - \frac{1}{2\tau_F} + \frac{1}{\tau_R} \frac{(1-v)(1+u-2v)}{F_1^2(u, v)} \right) \frac{\partial F_1}{\partial v} \\ & + \frac{1}{\tau_R} \frac{3+5u-8v}{F_1(u, v)} - \frac{\partial F_2}{\partial u} - \frac{1}{\tau_R} \left(3F_4(v) - (1-v) \frac{\partial F_4}{\partial v} \right) + \frac{5}{2\tau_R} = 0 \end{aligned}$$

we can solve the simpler

$$\begin{aligned} & \frac{2}{\tau_R} \frac{(1-u)(1+u-2v)}{\tilde{F}_1^2(u, v)} \frac{\partial \tilde{F}_1}{\partial u} + \frac{1}{\tau_R} \frac{(1-v)(1+u-2v)}{\tilde{F}_1^2(u, v)} \frac{\partial \tilde{F}_1}{\partial v} + \frac{1}{\tau_R} \frac{3+5u-8v}{\tilde{F}_1(u, v)} \\ & + \frac{2}{\tau_R} (1-\beta) = 0 \end{aligned} \quad (30)$$

where $\beta = (1-u^*)(1+u^*-2v^*)^{-1}$; the solution, $\tilde{F}_1(u, v)$, is an upper bound of $F_1(u, v)$, hence any function bounded by the solution $\tilde{F}_1(u, v)$ is a function for which the system is likely to have at least one invariant curve. Indeed, equation (18) is solvable to

$$\tilde{F}_1(u, v) = \frac{(1-u)^2(1+u-2v)^{1/3}}{2(\beta-1)(3-2u-v)(1+u-2v)^{1/3} + \Phi_1\left(\frac{u+v}{2}\right)}$$

where Φ_1 an arbitrary function of the characteristic $\frac{u+v}{2}$. In the original coordinates, it takes the form

$$\tilde{f}_1(x, z) = \frac{x^2(1-x-z)^{1/3}}{(\beta-1)(1+x-z)(1-x-z)^{1/3} + \varphi_1\left(z - \frac{x}{2}\right)} \quad (31)$$

where $\beta = x^*/y^*$ and φ_1 an arbitrary function of the characteristic $z - \frac{x}{2}$. For simple forms of φ_1 (constant, linear, quadratic, exponential, *etc.* over the characteristic) and $\beta > 1$,²³ this curve is

²³ The latter is not a necessary condition, however it secures that the function is similar to the saturation functions found in the SIRS literature. Moreover, this assumption means that $x^* > y^*$, which is generally reasonable.

positive for $x + z < 1$ and has a maximum, as expected. Hence, any similar function, $f_1(x, z)$, that is bounded by above from $\tilde{f}_1(x, z)$, allows for at least one invariant curve.

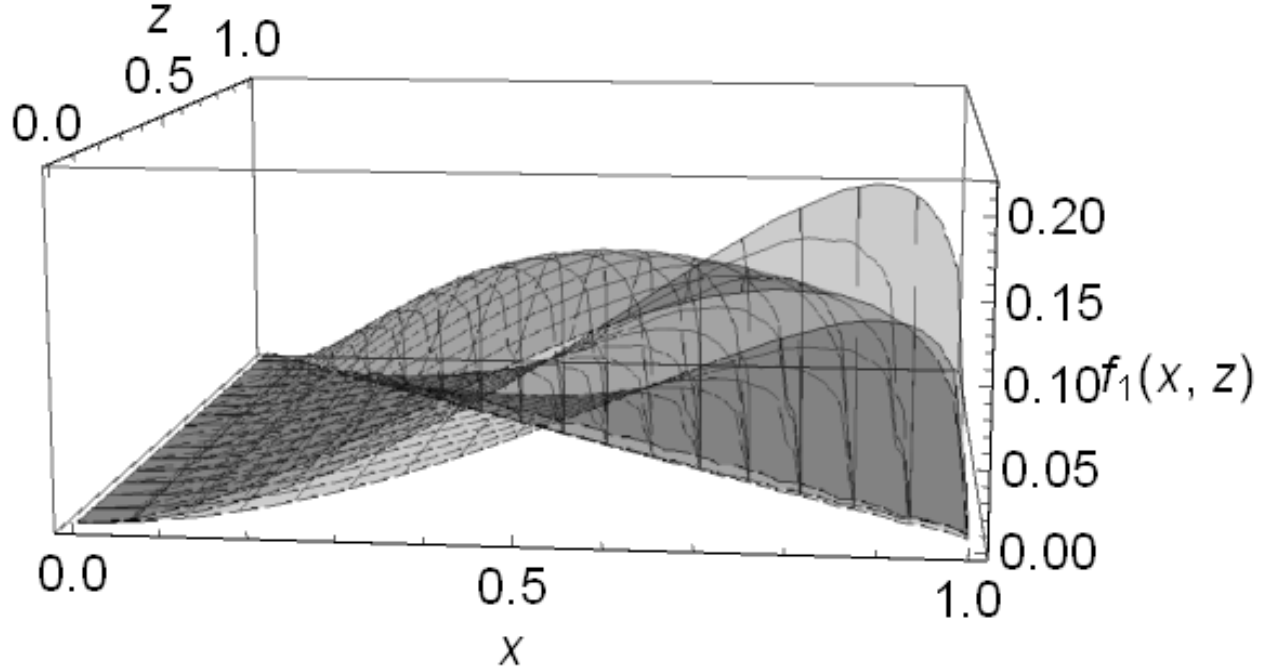


Figure 4: Some profiles for the function $\tilde{f}_1(x, z)$ in equation (31), which serve as upper bounds for the ‘financing function’ $f_1(x, z)$.

4.3. Global behavior

Combining the above, we can first of all discuss the equilibria.

- When $\tau_F > \tau_R$, there are three equilibrium points present: the ‘overaccumulation’ point, which is always unstable; the ‘excess money’ point, which is generally stable; and the ‘normal reproduction’ point, which is always stable (with oscillatory patterns about it). For the latter two to coexist on a two-dimensional plane, there must be some form of separatrix curve between them. So, the (reduced) phase space is divided to two subspaces; the solutions on the one side converge to the NR point, while the solutions on the other side converge to the EM point.
- When $\tau_F > \tau_R$, there are two equilibrium points present: the ‘excess commodities’ point, which is always stable; and the ‘normal reproduction’ point, which is always stable as well (with oscillatory patterns about it). Similarly, the two stable equilibrium points cannot

coexist, unless the (reduced) phase space is divided by a separatrix curve, with the solutions on the one side converging the NR point and the solutions on the other side converging to the EM point.

In either case, it is essential that there exists at least one curve that divides the phase space. This allows for the identification of the bifurcation occurring.

As noted by Wiggins (2013), there are saddle points whose stable and unstable manifolds connect, creating a closed loop known as a ‘homoclinic’ orbit; this is a unique orbit for the system that begins from the unstable ‘side’ of the saddle point and ends on its stable ‘side’. Such homoclinic orbits are separatrices for the phase space of the system, which enclose an equilibrium point, whose local asymptotic stability is unmatched with the asymptotic behavior of the system outside the loop. Moreover, given the smooth change of the system over the values of an exogenously given parameter, the homoclinic orbit may ‘break’, allowing for the emergence of a limit cycle; this is known as a ‘homoclinic bifurcation’ (Shilnikov 1968; Perko 1994).²⁴

Consequently, what we expect in both cases is a homoclinic orbit.

- When $\tau_F > \tau_R$, the homoclinic orbit begins and ends on the unstable OA equilibrium (solid curve in Figure 1a). Then, the NR equilibrium point undergoes a Bautin bifurcation, whereas a large stable and a small unstable limit cycle emerge (the solid and the dashed closed curves in Figure 1b, respectively), around the equilibrium point and within the homoclinic loop. Within the homoclinic orbit, the OA equilibrium is always a saddle and the NR equilibrium point is always a stable focus; while, outside, the EC equilibrium point is always stable.

²⁴ To prove there is a homoclinic bifurcation, we use Shilnikov’s theorem; namely, we check whether the sum of the real parts of the dominant positive and negative eigenvalues of the system around an equilibrium point changes sign. For details into this, see Appendix C.

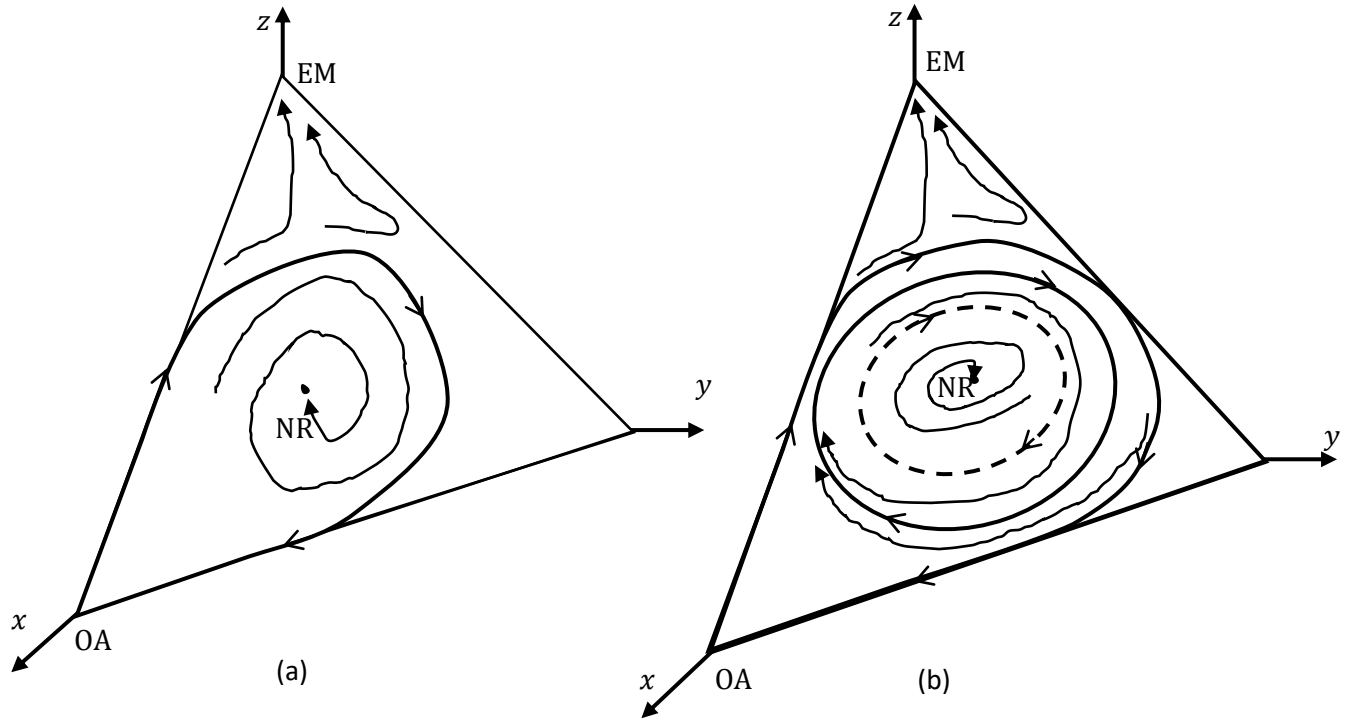


Figure 5: The phase space of the system for $\tau_F > \tau_R$, before the bifurcations (a), after the homoclinic bifurcation (b), where one unstable limit cycle emerges, and after the Bautin bifurcation (c), where a stable and an unstable limit cycle coexist.

- When $\tau_R > \tau_F$, the homoclinic orbit begins and ends on the EC point (solid curve in Figure 6a). Again, it undergoes a homoclinic bifurcation, being substituted by an unstable limit cycle (dashed curve in Figure 6b).

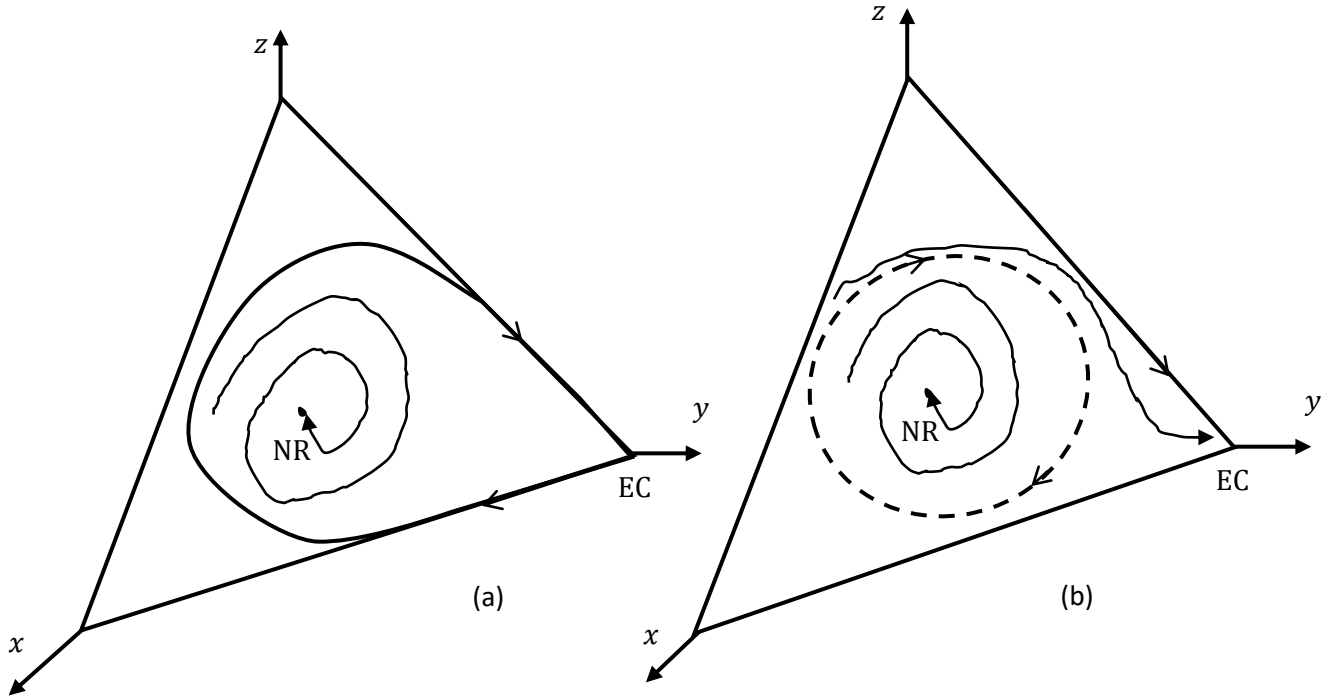


Figure 6: The phase space of the system for $\tau_R > \tau_F$, before (a) and after (b) the homoclinic bifurcation (b), where one unstable limit cycle emerges.

Thus, the overall behavior of the system is as follows: Either the ‘excess money’ or the ‘excess commodities’ crises are possible, depending on whether the financial lag is greater than the realization lag; both cases are secular stagnations, in the sense that solutions may converge to them from some region of the phase space and the system does not evolve beyond that. However, the ‘normal reproduction’ point is also possible and solutions may converge to it (by means of oscillations) from a different region of the phase space. The coexistence of both a ‘normal reproduction’ point and a stagnation point is possible only if the phase space is divided in regions. This is a case of ‘local stability vs. global instability’ as noted by Semmler *et al.* (2023). Essentially, the circuit of capital is stable and reduces to a generalized version of Marx’s Schemes of Expanded Reproduction if the perturbations about it are small; if, however, the perturbations are large enough, the system becomes unstable, either by resulting to permanent oscillations (the case of Figure 1c, for $\tau_F > \tau_R$), or by reduced to a secular stagnation in either ‘excess money’ (when $\tau_F > \tau_R$) or ‘excess commodities’ (when $\tau_R > \tau_F$) crises.

It is noteworthy that the type of possible stagnation is determined by the pace of turnover of either commercial or monetary capital. Basu (2014) also distinguishes different cases of expansion of the system depending on the magnitude of the financial lag; in his case, given an exogenously fixed minimum τ_R , when τ_F exceeds a threshold, the growth is considered wage-led, and when τ_F lies below a threshold, the growth is considered profit-led. Following his thought that τ_R is minimum, the change over τ_F reveals the following:

- When τ_F becomes small – that is, when money acts mainly as a means of transaction and the social capital does not need to spend significant time in this form, the system suffers from a possible underconsumption crisis, that will eventually lead to its ‘doom’ (as it cannot escape the EC point by any endogenous mechanism). This echoes the theories by Luxemburg, Moszkowska and Sweezy, where overproduction and the impoverishment of the masses leads to crises where surplus value fails to realize. However, unlike their theories, underconsumption in this case results as a ‘final state’ of the system and only when some of its forces have been ignored; thus, this situation is closer to Engels’ rejection of underconsumption crises as “a necessary condition of all forms of society based on exploitation [that] tells us just as little why crises exist today as why they did not exist before” (Engels 1877, pp. 340-341). Notably, the EC point may be unstable, allowing for damped oscillations towards the ‘normal reproduction’ point.
- When τ_F becomes sufficiently large – that is, when money takes a more active role in the system, *e.g.* as a means of hoarding, or as a means of regulator in savings and investment (financialization), the system no more suffers from an underconsumption crisis, but from a financialization stagnation, which similarly appears as its ‘doom’ (as the system cannot escape the EM point by itself). This echoes theories like Tugan-Baranovski’s and Hilferding’s, who emphasized the role of financialization in widening the ‘disproportions’ of the capitalist system and thus deepening its decennial cycles; but, more importantly, it echoes more modern theories proposing that the domination of the system by financial capital paves its way to deeper and more persistent crises. However, one should note again that the EM point indicates stagnation rather than recurring crises. In contrast, the unstable ‘overaccumulation’ point seems to be closely linked to the ‘normal reproduction’ one (the latter being enclosed in the homoclinic loop of the former), leading to damping oscillations generated by the rise of the organic composition of capital and the decline of the rate of

profit. Even further, a limit cycle may exist about the NR point (and within the aforementioned homoclinic loop), acting as a global attractor of the system, similar to the Kondratiev or the Kuznets cycles (see Chatzarakis *et al.* 2022).

In short, we can argue that both the possibility of ‘normal reproduction’ (as in Marx’s Schemes) and of recurring crises (as in Kondratiev’s long cycles) are present, especially when money is allowed to play all its possible roles in a capitalist system. Moreover, the cases of ‘type I’ and ‘type II’ crises, as well as third one, the ‘excess money’ or ‘financialization’ crises, may be considered. Notably, ‘type I’ and ‘type III’ may coexist, but not together with ‘type II’; the latter becomes possible when money acts merely as a means of transactions, while the former become important when money assumes all other functions. Furthermore, ‘type II’ (underconsumption) and ‘type III’ (financialization) crises appear mostly as secular tendencies leading to a ‘final state’, rather than generators of recurring patterns. In contrast, ‘type I’ (overaccumulation) crisis, although an unstable equilibrium *per se*, is vital in shaping both the local stability and the global instability in relation to the ‘normal reproduction’ point. Thus, we may conclude that both the case of (short-range) stability of expanded reproduction and the case of (long-range) instability in the form of recurring cycles depends largely on the influence of overaccumulation.

5. Conclusion

The circuit of capital is a scheme introduced by Marx with the purpose to analyze the transformation of capital from one form to the other as a whole; at the same time, it proved a powerful tool in analyzing the turnover time of capital, the nature of money and credit, the accumulation of capital and the reproduction of the system, as well as the possibility and actuality of different types of crises in the capitalist system.

Following Foley (1982, 1986a and 1986b), we attempted to model the circuit of capital as a dynamical system. But, unlike him, we focused on turning this to a closed and autonomous system, which can be qualitatively analyzed. The three variables in use were ratios of the rates of change of the stocks of productive, commercial and money capital, over the capitalized fraction of surplus value; these were proved sufficient to model the simple case, where no leakages are observed in the circuit.

Interestingly, the system resembled significant similarities to the SIRS epidemiological models, that also describe a circular transformation of the population (in our case, of the total capital) from one state (or form) to another. This resemblance is encouraging, as the epidemiological models have the behavior expected to be found here as well: trivial equilibrium points signifying stagnation, and (at least) one non-trivial one, signifying an endemic situation (normal reproduction of the system, in our case). The analysis of the system indeed revealed four such equilibrium points, the ‘overaccumulation’ stagnation point (signifying the ‘type I’ crises), the ‘excess commodities’ stagnation point (signifying the ‘type II’ crises), the ‘excess money’ stagnation point (signifying financialization, or a ‘type III’ crises) and the ‘normal reproduction’ point.

Of these four points, the fourth exists at all times and it is stable. The second exists when the financial lag is small and money acts mainly as a means of transactions; it is generally assumed stable, though it is possible it is a saddle. The first and third exist when the financial lag is large enough to allow for money to assume other functions (including money as commodity, as credit and as means of hoarding); of these, the first is always a saddle and the fourth is generally stable. What we uncovered is that, for either case, the system contains a separatrix dividing the phase space into regions where the ‘normal reproduction’ point is stable and regions where either the ‘excess money’ or the ‘excess commodities’ equilibria are stable. Thus, we may conclude that in such a system, there is the possibility of convergence to a case of normal expansion, similar to Marx’s Schemes of Expanded Reproduction; the possibility of recurrent cycles (as at least one limit cycle may emerge by means of bifurcations), that mimic the Kuznets or Kondratiev cycles; and the possibility of ‘type II’ (underconsumption) or ‘type III’ (financialization) breakdowns, which however take the form of secular stagnations.

It suffices to say that the analysis is not complete and several of these need to be clarified further. Since the assumed forms of the functions are well-justified to produce such a behavior, simulations could follow to further solidify the arguments.

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Appendix A: Asymptotic Stability Criteria for the Equilibria

To examine the local asymptotic stability of the equilibrium points, we consider small linear perturbations, Δx , Δy and Δz , around each of the equilibrium points and consider their evolution by means of the linearized system. If the real parts of all eigenvalues of the linearized system are negative, then the respective equilibrium point is stable, but if even one of them is positive, the respective equilibrium point is unstable; if the imaginary parts of the eigenvalues are zero, then the stability (attraction) or instability (repulsion) is monotonic, while if there are non-zero imaginary parts, then the stability (attraction) or instability (repulsion) is oscillatory.

Given the EM equilibrium point $(x^* = 0, y^* = 0, z^* = 1)$

$$\begin{bmatrix} \dot{\Delta x} \\ \dot{\Delta y} \\ \dot{\Delta z} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} - \frac{1}{\tau_R} + \frac{1}{\tau_F} \left(\frac{\partial f_1}{\partial x} \right)_{\{0,1\}} & 0 & \frac{1}{\tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,1\}} \\ \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} - \frac{1}{\tau_R} \left(\frac{\partial f_1}{\partial x} \right)_{\{0,1\}} & -\frac{2}{\tau_R} + \frac{1}{\tau_F} & -\frac{1}{\tau_R} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,1\}} \\ -\frac{\tau_R - \tau_F}{\tau_R \tau_F} \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} & \frac{1}{\tau_R} & -\frac{1}{\tau_R} + \frac{1}{\tau_F} - \frac{\tau_R - \tau_F}{\tau_R \tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,1\}} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

The characteristic polynomial of this matrix is

$$\begin{aligned} \lambda^3 + & \left[\frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} + \frac{1}{\tau_R} \left(4 - \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) - \frac{1}{\tau_F} \left(3 + \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \right] \lambda^2 \\ & + \left[\frac{1}{\tau_R^2} \left(5 - 2 \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) + \left(-\frac{2\tau_R - 3\tau_F}{\tau_P \tau_R} + \frac{\tau_R - \tau_F}{\tau_P \tau_R \tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} \right. \\ & - \frac{1}{\tau_R \tau_F} \left(8 + 3 \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - 5 \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \\ & \left. + \frac{1}{\tau_F^2} \left(3 + 2 \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - 2 \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \right] \lambda \\ & + \left[\left(\frac{2}{\tau_P \tau_R^2} + \frac{1}{\tau_P \tau_F^2} - \frac{3}{\tau_P \tau_R \tau_F} - \frac{(\tau_R - \tau_F)^2}{\tau_P \tau_R^2 \tau_F^2} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} \right. \\ & + \frac{1}{\tau_R^3} \left(2 - \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) - \frac{1}{\tau_R^2 \tau_F} \left(5 - 2 \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - 4 \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \\ & \left. + \frac{1}{\tau_R \tau_F^2} \left(4 + 3 \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - 4 \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) - \frac{1}{\tau_F^3} \left(1 - \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \right] = 0 \end{aligned}$$

Using the Routh-Hurwitz criterion, we can prove that all eigenvalues are negative; a closer look reveals that, for economically meaningful values of the lags τ_P , τ_R and τ_F (such that $\tau_F > \tau_R$ so that the point exists) and of the derivatives, the eigenvalues are real. More particularly, the first eigenvalue (which is $(\tau_F - \tau_R)/(\tau_R \tau_F)$) is always negative, while the second and third ones are real when

$$\left(1 - \frac{\tau_R}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}}\right)^2 + \left(\frac{\tau_R}{\tau_F} \left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}} - \frac{\tau_R + \tau_F}{\tau_F} \left(\frac{\partial f_1}{\partial z}\right)_{\{0,0\}}\right)^2 + 2 \left(1 - \frac{\tau_R}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}}\right) \left(\frac{\tau_R}{\tau_F} \left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}} - \frac{\tau_R + \tau_F}{\tau_F} \left(\frac{\partial f_1}{\partial z}\right)_{\{0,0\}}\right) > 0$$

and negative when

$$\frac{\tau_P}{2\tau_F} \left(\frac{3\tau_F - 2\tau_R}{\tau_R} + \left(\frac{\partial f_1}{\partial x}\right)_{\{0,1\}}\right) + 2 \frac{\tau_F - \tau_R}{\tau_R} \left(\frac{\partial f_1}{\partial z}\right)_{\{0,1\}} < \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}}$$

both of which hold. Thus, the EM equilibrium point is always monotonically stable, which means that an ‘excess money’ crisis is a secular stagnation.

Given the EC equilibrium point ($x^* = 0, y^* = 1, z^* = 0$)

$$\begin{bmatrix} \dot{\Delta x} \\ \dot{\Delta y} \\ \dot{\Delta z} \end{bmatrix} = \begin{bmatrix} j_{11} & 0 & \frac{1}{\tau_F} \left(\frac{\partial f_1}{\partial z}\right)_{\{0,0\}} \\ j_{21} & -\frac{2}{\tau_F} & \frac{\tau_R - 2\tau_F}{\tau_F^2} \left(\frac{\partial f_1}{\partial z}\right)_{\{0,0\}} \\ -\frac{\tau_R - \tau_F}{\tau_F^2} \left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}} & \frac{1}{\tau_F} & j_{33} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

Where

$$j_{11} = -\frac{1}{\tau_F} \left(1 - \left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}}\right) - \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}}$$

$$j_{21} = \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{0\}} + \frac{\tau_R - 2\tau_F}{\tau_F^2} \left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}}$$

$$j_{33} = -\frac{\tau_R - \tau_F}{\tau_F^2} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} - \frac{1}{\tau_F}$$

The characteristic polynomial of this matrix is

$$\begin{aligned} \lambda^3 + & \left[\frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} + \frac{4}{\tau_F} - \frac{1}{\tau_F} \left(\left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} - \frac{\tau_R - \tau_F}{\tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \right] \lambda^2 \\ & + \left[\left(\frac{1}{\tau_P \tau_F} \left(3 + \frac{\tau_R - \tau_F}{\tau_F} \right) \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} + \frac{5}{\tau_F^2} \right. \\ & \left. - \frac{1}{\tau_F^2} \left(3 \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} + \frac{2\tau_R - \tau_F}{\tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \right] \lambda \\ & + \left[\left(\frac{2}{\tau_P \tau_F^2} + \frac{\tau_R - \tau_F}{\tau_P \tau_F^3} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{0\}} + \frac{\tau_R}{\tau_F^3} \left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} - \frac{2}{\tau_F^3} \left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} \right] = 0 \end{aligned}$$

Hence, the first eigenvalues of the system is always real and negative (equal to $-\tau_F^{-1}$). Using the Routh-Hurwitz criterions, we can prove that the other two are

- complex with negative real parts for small values of the derivatives of $f_1(x, z)$,
- real and negative for larger values of the derivatives of $f_1(x, z)$, and
- real, one being positive and one negative, for even large values of the derivatives of $f_1(x, z)$.

Thus, the stability of the EC point changes as the shape of $f_1(x, z)$ changes; the smoother and flatter it is around $x^* = z^* = 0$, the EC point is a stable focus (or a stable node), and the steeper it becomes, the EC point becomes a saddle. We must note that the flatness or the steepness of the function is highly related to the position of its maximum. If the maximum is closer to $x^* = z^* = 0$, then the function is expected to be steep in both direction – if the maximum is closer to either $x^* = 0$, then $\left(\frac{\partial f_1}{\partial x} \right)_{\{0,0\}} > 1$ and $\left(\frac{\partial f_1}{\partial z} \right)_{\{0,0\}} < 1$ and the function will be steep in the direction of x and flat in the direction of z (and *vice versa* if it is closer to $z^* = 0$). As a result, the shift of the NR equilibrium point (and any similar change in the shape of the ‘production function’) will affect the stability of the EC point, making it unstable as the points would come closer.

Given the OA equilibrium point ($x^* = 1, y^* = z^* = 0$)

$$\begin{bmatrix} \dot{\Delta x} \\ \dot{\Delta y} \\ \dot{\Delta z} \end{bmatrix} = \begin{bmatrix} j_{11} & \frac{1}{\tau_R} - \frac{1}{2\tau_R - \tau_F} & \frac{1}{\tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \\ \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} - \frac{1}{\tau_R} \left(\frac{\partial f_1}{\partial x} \right)_{\{1,0\}} & -\frac{2}{\tau_R} & -\frac{1}{\tau_R} \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \\ -\frac{\tau_R - \tau_F}{\tau_R \tau_F} \left(\frac{\partial f_1}{\partial x} \right)_{\{1,0\}} & \frac{1}{2\tau_R - \tau_F} & j_{33} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

where

$$j_{11} = -\frac{1}{\tau_R} - \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} + \frac{1}{\tau_F} \left(\frac{\partial f_1}{\partial x} \right)_{\{1,0\}}$$

$$j_{33} = -\frac{1}{\tau_R} + \frac{\tau_R - \tau_F}{\tau_R \tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}}$$

The characteristic polynomial of this matrix is

$$\begin{aligned} \lambda^3 &+ \left[\frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} + \frac{4}{\tau_R} - \frac{1}{\tau_F} \left(\left(\frac{\partial f_1}{\partial x} \right)_{\{1,0\}} - \frac{\tau_R - \tau_F}{\tau_R} \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \right) \right] \lambda^2 \\ &+ \left[\frac{2}{\tau_R^3} + \frac{1}{\tau_P \tau_R (2\tau_R - \tau_F)} \left(3 - \frac{\tau_F}{\tau_R} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} \right. \\ &+ \frac{1}{\tau_R^3} \left(1 - \frac{2\tau_R}{\tau_F} - \frac{\tau_R}{2\tau_R - \tau_F} \right) \left(\frac{\partial f_1}{\partial x} \right)_{\{1,0\}} - \frac{1}{\tau_R^3} \left(2 - \frac{2\tau_R}{\tau_F} - \frac{\tau_R}{2\tau_R - \tau_F} \right) \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \\ &+ \left. \frac{\tau_R - \tau_F}{\tau_P \tau_R^2 \tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} \right] \lambda \\ &+ \left[\frac{5}{\tau_R^2} + \frac{1}{\tau_P (2\tau_R - \tau_F)} \left(5 - \frac{2\tau_F}{\tau_R} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} \right. \\ &+ \frac{1}{\tau_R^2} \left(1 - \frac{3\tau_R}{\tau_F} - \frac{\tau_R}{2\tau_R - \tau_F} \right) \left(\frac{\partial f_1}{\partial x} \right)_{\{1,0\}} - \frac{1}{\tau_R^2} \left(3 - \frac{3\tau_R}{\tau_F} - \frac{\tau_R}{2\tau_R - \tau_F} \right) \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \\ &+ \left. \frac{\tau_R - \tau_F}{\tau_P \tau_R \tau_F} \left(\frac{\partial f_1}{\partial z} \right)_{\{1,0\}} \left(\frac{\partial f_2}{\partial x} \right)_{\{1\}} \right] = 0 \end{aligned}$$

Hence, the first eigenvalue is real and negative (equal to $-\tau_R^{-1}$). Using the Routh-Hurwitz criterions, we can prove that the other two are real because

$$\begin{aligned}
& \frac{4\tau_R(\tau_R - \tau_F)}{(2\tau_R - \tau_F)^2} - 2\left(2 - \frac{\tau_R}{\tau_F} - \frac{2\tau_R}{2\tau_R - \tau_F}\right)\left(\frac{\partial f_1}{\partial x}\right)_{\{1,0\}} + 2\left(1 - \frac{\tau_R}{\tau_F} - \frac{2\tau_R}{2\tau_R - \tau_F}\right)\left(\frac{\partial f_1}{\partial z}\right)_{\{1,0\}} \\
& + \left(\frac{\tau_F}{2\tau_R - \tau_F} - \frac{\tau_R}{\tau_P}\left(\frac{\partial f_2}{\partial x}\right)_{\{1\}}\right)^2 + \left(\frac{\tau_R}{\tau_F}\left(\frac{\partial f_1}{\partial x}\right)_{\{1,0\}} + \frac{\tau_F - \tau_R}{\tau_F}\left(\frac{\partial f_1}{\partial z}\right)_{\{1,0\}}\right)^2 \\
& + 2\frac{\tau_R}{\tau_P}\left(\frac{\tau_R}{\tau_F}\left(\frac{\partial f_1}{\partial x}\right)_{\{1,0\}} + \frac{\tau_F - \tau_R}{\tau_F}\left(\frac{\partial f_1}{\partial z}\right)_{\{1,0\}}\right)\left(\frac{\partial f_2}{\partial x}\right)_{\{1\}} > 0
\end{aligned}$$

holds, and at least one of them is positive when $\left(\frac{\partial f_1}{\partial x}\right)_{\{1,0\}}$ is sufficiently small (so $f_1(x, z)$ is flat towards $x^* = 1$). The result becomes even stronger when $\left(\frac{\partial f_1}{\partial z}\right)_{\{1,0\}}$ is large (so $f_1(x, z)$ is also steep towards $z^* = 0$). These mean that the OA equilibrium point is a saddle when the maximum of $f_1(x, z)$ is closer to $z^* = 0$; or, equivalently, the accumulation of monetary capital is relatively low in the NR equilibrium point.

Given the NR equilibrium,

$$\begin{bmatrix} \dot{\Delta x} \\ \dot{\Delta y} \\ \dot{\Delta z} \end{bmatrix} = \begin{bmatrix} j_{11} & -\frac{1}{\tau_R} \frac{x^*}{f_1(x^*, z^*)} & 0 \\ \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{x^*\}} & j_{22} & 0 \\ 0 & \frac{1}{\tau_R} \left(\frac{x^* + y^*}{f_1(x^*, z^*)} + 1\right) & j_{33} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

where

$$\begin{aligned}
j_{11} &= -\frac{1}{\tau_R} \left(1 + \frac{y^*}{f_1(x^*, z^*)}\right) - \frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{x^*\}} + \frac{1}{\tau_F} f_4(z^*) \\
j_{22} &= -\frac{2}{\tau_R} \left(\frac{y^*}{f_1(x^*, z^*)} + 1\right) + \frac{1}{\tau_F} f_4(z^*) \\
j_{33} &= -\frac{1}{\tau_R} \left(1 + \frac{y^*}{f_1(x^*, z^*)}\right) + \frac{1}{\tau_F} f_4(z^*)
\end{aligned}$$

The characteristic polynomial of this matrix

$$\begin{aligned}
& \lambda^3 + \left[\frac{1}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} + \frac{4}{\tau_R} \left(1 + \frac{y^*}{f_1(x^*, z^*)} \right) - \frac{3}{\tau_F} f_4(z^*) \right] \lambda^2 \\
& + \left[\frac{5}{\tau_R^2} \left(1 - \frac{2y^*}{f_1(x^*, z^*)} + \frac{y^{*2}}{f_1^2(x^*, z^*)} \right) + \frac{1}{\tau_P \tau_R} \left(3 + \frac{x^* + 3y^*}{f_1(x^*, z^*)} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} \right. \\
& - \frac{2}{\tau_P \tau_F} f_4(z^*) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} - \frac{8}{\tau_R \tau_F} \left(1 + \frac{y^*}{f_1(x^*, z^*)} \right) f_4(z^*) + \frac{3}{\tau_F^2} f_4^2(z^*) \left. \right] \lambda \\
& + \left[\frac{1}{\tau_P \tau_R^2} \left(2 + \frac{x^* + 4y^*}{f_1(x^*, z^*)} + \frac{y^*(x^* + 2y^*)}{f_1^2(x^*, z^*)} \right) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} + \frac{1}{\tau_P \tau_F^2} f_4^2(z^*) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} \right. \\
& - \frac{5}{\tau_R^2 \tau_F} \left(1 + \frac{2y^*}{f_1(x^*, z^*)} + \frac{y^{*2}}{f_1^2(x^*, z^*)} \right) f_4(z^*) - \frac{4}{\tau_R \tau_F^2} \left(1 + \frac{y^*}{f_1(x^*, z^*)} \right) f_4^2(z^*) \\
& - \frac{1}{\tau_P \tau_R \tau_F} \left(3 + \frac{x^* + 3y^*}{f_1(x^*, z^*)} \right) f_4(z^*) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} + \frac{1}{\tau_R \tau_F^2} f_4^2(z^*) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} \\
& \left. + \frac{2}{\tau_R^3} \left(1 + \frac{3y^*}{f_1(x^*, z^*)} + \frac{3y^{*2}}{f_1^2(x^*, z^*)} + \frac{3y^{*3}}{f_1^3(x^*, z^*)} \right) - \frac{1}{\tau_F^3} f_4^3(z^*) \right] = 0
\end{aligned}$$

Hence, the first eigenvalues is

$$-\frac{1}{\tau_R} \left(1 + \frac{y^*}{f_1(x^*, z^*)} \right) + \frac{1}{\tau_F} f_4(z^*)$$

and is negative when

$$1 + \frac{y^*}{f_1(x^*, z^*)} > \frac{\tau_R}{\tau_F} f_4(z^*)$$

which is usually the case, as the term in the l.h.s. is greater than 1 (note that it is 2 when the condition $y^* = f_1(x^*, z^*)$ holds) and the term in the r.h.s. is smaller than 1 (as $\tau_R < \tau_F$ and $f_4(z) \leq 1$). As for the other two eigenvalues, they are

$$\begin{aligned}
\lambda_{2,3} = & -\frac{3}{2\tau_R} \left(1 + \frac{y^*}{f_1(x^*, z^*)} \right) + \frac{1}{\tau_F} f_4(z^*) - \frac{2}{\tau_P} \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}} \\
& \pm \sqrt{\frac{\tau_P^2 (y^* + f_1(x^*, z^*))^2 + \tau_R \left(\tau_R f_1(x^*, z^*) - 2\tau_P (2x^* + y^* + f_1(x^*, z^*)) \right) f_1(x^*, z^*) \left(\frac{\partial f_2}{\partial x} \right)_{\{x^*\}}^2}{2\tau_P \tau_R f_1(x^*, z^*)}}
\end{aligned}$$

that can have imaginary parts if

$$\frac{\tau_P}{\tau_R} \left(1 + \frac{y^*}{f_1(x^*, z^*)}\right)^2 + \frac{\tau_R}{\tau_P} \left(\frac{\partial f_2}{\partial x}\right)_{\{x^*\}}^2 < 2 \left(1 + \frac{2x^* + y^*}{f_1(x^*, z^*)}\right) \left(\frac{\partial f_2}{\partial x}\right)_{\{x^*\}}^2$$

and whose real parts are negative if

$$\frac{3}{2} \left(1 + \frac{y^*}{f_1(x^*, z^*)}\right) + 2 \left(\frac{\partial f_2}{\partial x}\right)_{\{x^*\}} > \frac{\tau_R}{\tau_F} f_4(z^*)$$

which is usually the case, as the term in the l.h.s. is generally greater than 1 and the term in the r.h.s is always smaller than 1. As a result, the NR equilibrium is always a stable focus (whether we assume that $y^* = f_1(x^*, z^*)$ holds, as we did in the main text, or not).

Appendix B: Global Stability and Limit Cycle

Another thing we can consider, though, is the existence of a limit cycle. To determine whether a limit cycle exists in a three-dimensional system, we rely on a technique developed by Li and Muldowney (1993). From this, we get three conditions as lower bounds,

$$\begin{aligned} & (x - 3y)f_1(x, z) + y(x + y) \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial z} \right) \\ & > \left(3 + \frac{\tau_R}{\tau_F} \left(2f_4(z) + (1 - z) \frac{\partial f_4}{\partial z} \right) - \left(1 - \frac{\tau_R}{\tau_F} \right) \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial z} \right) \right) f_1^2(x, z) \end{aligned}$$

$$\begin{aligned} & (x - 2y)f_1(x, z) + xy \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial z} \right) \\ & > \left(2 + \frac{\tau_R}{\tau_F} \left(2f_4(z) - \frac{\partial f_4}{\partial z} \right) - \frac{\partial f_2}{\partial x} + \frac{\tau_R}{\tau_F} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial z} \right) \right) f_1^2(x, z) \end{aligned}$$

$$\begin{aligned} & (2x - y)f_1(x, z) - y \left(y \frac{\partial f_1}{\partial x} + (2x + y) \frac{\partial f_1}{\partial z} \right) \\ & > \left(1 + \frac{\tau_R}{\tau_F} \left(2f_4(z) - (2x + y) \frac{\partial f_4}{\partial z} \right) + \frac{\tau_R}{\tau_P} \frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial x} + \left(1 - 2 \frac{\tau_R}{\tau_F} \right) \frac{\partial f_1}{\partial z} \right) f_1^2(x, z) \end{aligned}$$

and three more conditions as upper bounds,

$$\begin{aligned}
& (x - 3y)f_1(x, z) + y(x + y) \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial z} \right) \\
& > \left(3 + \frac{\tau_R}{\tau_F} \left(2f_4(z) + (x + y) \frac{\partial f_4}{\partial z} \right) + \left(1 - \frac{\tau_R}{\tau_F} \right) \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial z} \right) \right) f_1^2(x, z) \\
& (x - 2y)f_1(x, z) + y(x + 2y) \frac{\partial f_1}{\partial x} - xy \frac{\partial f_1}{\partial z} \\
& > \left(2 + \frac{\tau_R}{\tau_F} \left(2f_4(z) - (x + 2y) \frac{\partial f_4}{\partial z} \right) + \frac{\tau_R}{\tau_F} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial z} \right) - \frac{\tau_R}{\tau_P} \frac{\partial f_2}{\partial x} \right) f_1^2(x, z) \\
& (2x - y)f_1(x, z) + y(2x + y) \frac{\partial f_1}{\partial x} - y^2 \frac{\partial f_1}{\partial z} \\
& > \left(-1 + \frac{\tau_R}{\tau_F} \left(2f_4(z) - (2x + y) \frac{\partial f_4}{\partial z} \right) + \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial z} + \frac{\tau_R}{\tau_P} \frac{\partial f_2}{\partial x} \right) f_1^2(x, z)
\end{aligned}$$

We proceed by combining the three conditions in each case as follows: we add the first two and subtract the third, obtaining the following two differential inequalities

$$\begin{aligned}
& (y^2 - f_1^2(x, z)) \frac{\partial f_1}{\partial x} > 2y f_1(x, z) + \left(2 + \frac{\tau_R}{\tau_F} f_4(z) - \frac{\tau_R}{\tau_P} \frac{\partial f_2}{\partial x} \right) f_1^2(x, z) \\
& y^2 \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial z} \right) > 2y f_1(x, z) + \left(\frac{5}{2} - \frac{\tau_R}{\tau_F} \left(f_4(z) - (1 - z) \frac{\partial f_4}{\partial z} \right) - \frac{\tau_R}{\tau_P} \frac{\partial f_2}{\partial x} \right) f_1^2(x, z)
\end{aligned}$$

Given that $y = 1 - x - z$ and following some simple specifications for $f_2(x)$ and $f_4(z)$, these two can be solved to yield a lower and an upper bound for $f_1(x, z)$ respectively, for which the system of equations (19)-(22) has at least one limit cycle. It is proved that, for $\tau_P > \tau_R, \tau_F$ the upper bound is generally negative, thus the conditions for $f_1(x, z)$ are not preserved. However, for $\tau_P < \tau_F < \tau_R$ (the case where the ‘excess commodities’ breakdown is possible), the upper bound is always positive and a limit cycle may exist; the same can be true for $\tau_P < \tau_R < \tau_F$ (the case where the ‘overaccumulation’ and ‘excess money’ equilibria are present), however the production lag should be much smaller than the other two, even smaller than 1. Further specification of these limit cycles is not possible. Yet, given the local asymptotic stability of the equilibrium point NR, from Bendixson’s criterion²⁵ it follows that such a limit cycle is likely to be unstable.

²⁵ According to Bendixson’s criterion, a limit cycle formed around an equilibrium point must have the opposite characterization from it; hence, the limit cycle of our model, that encloses a stable equilibrium point, must be unstable.

Appendix C: Homoclinic and Bautin Bifurcations

Finally, we will consider the case of homoclinic bifurcations for the EC equilibrium point, that is, the bifurcation that transforms a homoclinic orbit to a limit cycle; and of Bautin bifurcations for the NR equilibrium point, that is, the bifurcation that creates multiple limit cycles.

The former can be found using Shilnikov's (1968) criterion, which is to check the sign of the sum of the real parts of the leading stable and the leading unstable eigenvalues, that is, of the largest negative and the smallest positive real parts among the eigenvalues. It is easy to prove that for high values of $\left(\frac{\partial f_1}{\partial x}\right)_{\{0,0\}}$ and relatively low values of $\left(\frac{\partial f_1}{\partial z}\right)_{\{0,0\}}$ (that is, when the function is steep along the direction of x , but not as steep in the direction of z near the origin, or that its maximum is closer to $x^* = 0$ and not to $z^* = 0$), the afore-mentioned variable becomes positive, signifying the passing from a homoclinic orbit to a limit cycle. Given the nature of the maximum of $f_1(x, z)$, this transition can happen only when the accumulation of monetary capital is (slightly) faster than the accumulation of production capital in the NR equilibrium.

A similar transition does not occur in the case of the other two equilibrium points.

Concerning the Bautin transformation, it can be checked through the sign of two Lyapunov coefficients constructed from the higher derivatives of the system – essentially, from the coefficients of the fifth-order Taylor expansion of the r.h.s. of our differential equations (Kuznetsov 1998, ch. 8). This method is rather complicated and its results are not easy to present. However, the following simplifications are possible to demonstrate the results: we can assume that the higher-than-third-order derivatives of $f_1(x, z)$, $f_2(x)$ and $f_4(z)$ functions are negligible, since, in the case of the former two, the functions are smooth and strictly concave around the NR equilibrium – and, in the case of the latter, because the function resembles two logistic curves joined at the NR equilibrium. Then, given very small values of $\left(\frac{\partial^2 f_1}{\partial x \partial z}\right)_{\{x^*, z^*\}}$, there exist always points (x^*, y^*, z^*) for which the Lyapunov coefficient changes sign, thus giving rise to a double limit cycle.

