Heterogeneity as a sufficient Condition for Clustering of Defaults

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Abstract

This paper provides a theoretical model which highlights the role of heterogeneity of information in the emergence of temporal aggregation (clustering) of defaults in a leveraged economy. We show that the degree of heterogeneity plays a critical role in the persistence of the correlation between defaults in time. Specifically, a high degree of heterogeneity leads to an autocorrelation of the time sequence of defaults characterised by a hyperbolic decay rate, such that the autocorrelation function is not summable (infinite memory) and defaults are clustered. Conversely, if the degree of heterogeneity is reduced the autocorrelation function decays exponentially fast, and thus, correlation between defaults is only transient (short memory). Our model is also able to reproduce stylized facts, such as clustered volatility and non-Normal returns. Our findings suggest that future regulations might be directed at improving publicly available information, reducing the relative heterogeneity.

Keywords: Hedge funds; survival statistics; systemic risk; clustering

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1 Introduction

The hedge fund (HF) industry has experienced an explosive growth in recent years. The total size of the assets managed by HFs today stands at US$2.13 trillion (**?**). Due to the increasing weight of HFs in the financial market, failures of HFs can pose a major threat to the stability of the global financial system. The default of a number of high profile HFs, such as LTCM and HFs owned by Bear Stearns (**?**), testifies to this. At the same time, poor performance of HFs—a prerequisite for the failure of a HF—is empirically found to be strongly correlated across HFs (**?**), a phenomenon known as "contagion". The findings of **?** support the theoretical predictions of **?**, who provide a mechanism revealing how liquidity shocks can lead to downward liquidity spirals and thus to contagion[1]. The presence of contagion in the HF industry increases its potential to destabilize the financial market, contributing to systemic risk. Our work takes a step forward in this direction, by establishing a direct link between the heterogeneity in the quality of information about the fundamentals and a stronger form of contagion identified with the temporal aggregation of defaults, i.e. clustering of defaults. Using the definition of **?** clustering is determined by the divergence of the sum (or integral in continuous time) of the autocorrelation function of the default time sequence, and therefore, the presence of infinite memory in the underlying stochastic process describing the occurrence of defaults.

In order to study the correlation of default events in time, we develop a simple dynamic model with a representative noise trader and heterogeneous risk averse HFs which trade a risky asset. In the absence of HFs the demand of the noise trader leads the price of the asset to revert to its fundamental value. The HFs know the fundamental value of the asset, but each one with a different precision. Consequently, when the price of the asset is lower than its expected value, all HFs take a long position, but the difference in precision and risk aversion leads them to different levels of optimal demand. The HFs are liquidated (default) when their returns are low and their wealth drops under a given threshold. Furthermore, this heterogeneous demand behaviour leads to a different expected value of the waiting times between defaults across HFs. We assume, and later verify, that each HF defaults, on average, at a constant rate, which depends on the quality of information available to it. Consequently, the default of each HF follows a Poisson process. We show (Theorem 1) that the aggregation (mixing) of these elementary processes, in the limit of a continuum of HFs, leads to qualitatively different statistics on the aggregate level. Specifically, we show that the aggregate PDF exhibits a heavy-

[1]Other works which study the the causes of contagion in financial markets include **??**, **?** and **?**.

tail and thus, is scale-invariant (self-similar), a well-known characteristic of power-law behaviour. Furthermore, we prove that if the variance of the aggregate PDF diverges, then due to the self-similarity property of the aggregate PDF and the non-negligible probability assigned to long waiting-times, the time series of defaults after aggregation possesses infinite memory and therefore defaults are clustered (Theorem 2). We also solve the model numerically in order to test the validity of our theoretical assumptions about the Poisson character of defaults on the microscopic level and to study the properties of the aggregate PDF in a market with a finite number of traders. The simulations indeed vindicate the assumption of defaults occurring according to a Poisson process, when each HF is studied individually. We also establish that the intensity (expected rate of defaults) of the Poisson process, describing the default of each of the HFs, is a function of the quality of information about the fundamental value of the risky asset. Finally, numerical simulations attest to the fact that for a high degree of difference in the quality of information among HFs, the variance of the aggregate PDF tends to infinity. In this case, the comparison between the theoretical prediction of the asymptotic behaviour of the autocorrelation function of defaults and the numerical findings, reveals that our theoretical predictions are valid even in a market with a finite number of HFs and the clustering of defaults is confirmed. Our results show that the extent of heterogeneity, quantified by the difference in the precision of the information available between the best and worst informed HF, is a determinant factor for the clustering of defaults, and as such, the presence of systemic risk. The latter suggests that future regulations should also aim to increase publicly available information, in order to help decrease the heterogeneity in the behaviour of HFs.

The mechanism that leads to contagion is closely related to the "Leverage cycle" introduced by **??**, which consists in the pro-cyclical increase and decrease of leverage, due to the interplay between equity volatility and leverage. Even though this insight is useful for understanding the mechanism leading to contagion, in our model we are able to highlight the critical role of heterogeneity in determining the temporal structure of defaults, which is only of secondary importance in the works of Geanakoplos. Specifically, **?** argues that:

> ...It is interesting to observe that the kind of heterogeneity influences the amount of equilibrium leverage, and hence equilibrium asset prices, and equilibrium default. ...Thus, when the heterogeneity stems entirely from one-dimensional differences in opinion, equilibrium leverage entails no default...

Also in ? the author continues,

> ... Of course the asymmetric information revolution in economics was a tremendous advance, and asymmetric information plays a critical role in many lender-borrower relationships; sometimes, however, the profession becomes obsessed with it... In my model, the only thing backing the loan is the physical collateral. Because the loans are no-recourse loans, there is no need to learn anything about the borrower. All that matters is the collateral.

This framework has been extended, allowing the study of the Leverage cycle for an arbitrary number of periods in two recent papers by ? and ?. ? argue that the leverage cycle, in this expanded framework, is the driving force for the emergence of fat tails in the return distribution and clustered volatility, while ? study the effectiveness of regulations under the Basel II accord. The general setup of our model is closely related to these two papers, as it focuses on the effects of heterogeneity in HF behaviour in a leveraged economy. However, it differs with the aforementioned papers in two ways: first, in our model, HFs are represented by risk-averse agents, who maximise a function of their expected rate of profit, and, moreover, the demand heterogeneity stems from the different quality of information available to each HF concerning the fundamental value of the risky asset. In contrast, in ? and ? the behaviour of the HFs is not micro-founded, HFs are risk-neutral, and the heterogeneity is related to an "aggression parameter", which is assumed to differ across HFs. Secondly, while in ? and ? HFs receive funds from two sources (loans from a bank and from fund investors), in our model funds only come in the form of loans offered by a bank.

The structure of the rest of the paper is as follows. Section 2 presents the economic framework that we use. In Sec. 3, we analyse the theoretical predictions of our model and Sec. 4 discusses the numerical results and points out regulatory implications. Finally, Sec. 5 provides a short summary with concluding remarks.

2 Model

We study an economy with two assets, one risk-free (cash C) and one risky, two types of agents and a bank. The risky asset exists in a finite quantity equal to N and can be viewed as a stock. These two assets are held by a representative noise trader and K types of hedge funds (HFs). Finally, there is also a bank which extends loans to the HFs, using the risky asset owned by the HFs as collateral[2].

The demand d^{nt} of the representative mean-reverting noise-trader for the risky asset, in terms of cash value, is assumed to follow a first-order autoregressive [AR(1)] process (**???**).

$$\log d_t^{\mathrm{nt}} = \rho \log d_{t-1}^{nt} + \chi_t + (1-\rho)\log(VN). \tag{1}$$

$\rho \in (0,1)$ is a parameter controlling the rate of reverting to the mean[3], $\chi_t \sim \mathcal{N}\left(0, \sigma_{\mathrm{nt}}^2\right)$, and V is the fundamental value of the risky asset.

HFs are represented by risk-averse agents, whose performance depends on their rate of return, a standard measure of performance in the HF industry. For example, Fig. 1 shows the ranking of the 100 top large HFs as reported by **?**[4].

The utility function of HF j is given by

$$U(r_t^j) = 1 - e^{-\alpha r_t^j} \equiv 1 - e^{-\alpha (W_t^j - W_{t-1}^j)/W_{t-1}^j}, \tag{2}$$

where r_t^j is the rate of return of HF j over the course of a single time period, W_t^j is the wealth of the same HF at time t and $\alpha > 0$ is the Arrow-Pratt-De Finetti measure of relative risk aversion. In this way, the HFs have constant relative risk aversion (CRRA) and their absolute risk aversion is decreasing with increasing wealth (DARA). Hence, in order for a HF to increase its rate of return it turns to a riskier behaviour the higher its wealth is, for given investment opportunities.

The HFs are initially endowed with the amount of risk-free asset W_0. They are assumed to have information about the fundamental value of the risky asset and invest into it by taking advantage of the mispricing caused by the noise trader's behaviour.

Specifically, HF j receives a private noisy signal $\tilde{V}_j = V + \epsilon_j$ about the fundamental value of the risky asset, where the noise term ϵ_j is assumed to follow a Normal distribution with mean 0 and variance σ_j^2. Here the source of heterogeneity has to do with the difference in the quality of information about the fundamental value of the risky asset available to each HF, quantified by $s_j = 1/{\sigma_j}^2$. Otherwise, the HFs are identical, given that they are all characterised by the same coefficient of risk aversion. When the demand

[2]Herein the risk-free interest rate as well as the risk premium on lending is set to 0. We also assume that the bank is infinitely liquid.

[3]An AR(1) process of the form $x_t = \delta + \rho x_{t-1} + \chi_t$, with $\delta \in \mathbb{R}$, $|\rho| < 1$ and $\chi_t \sim \mathcal{N}(0, \sigma^2)$ is characterized by: $\mathbb{E}[x_t] = \delta/(1-\rho)$, $\mathrm{Var}[x_t] \equiv \mathbb{E}[(x_t - \delta/(1-\rho))^2] = \sigma^2/(1-\rho^2)$, while the normalised auto-covariance function is $\mathrm{Cov}[x_t, x_{t+s}]/\mathrm{Var}[x_t] = \rho^s$ and thus, for $\rho \in (0,1)$ the auto-covariance is a decreasing function of the lag variable s (**?**, pp. 30-31). Given that the expected value of x_t and the auto-covariance function are time-independent, the stochastic process is wide-sense stationary.

[4]Returns are for the first 10 months in 2013 ended on Oct. 31.

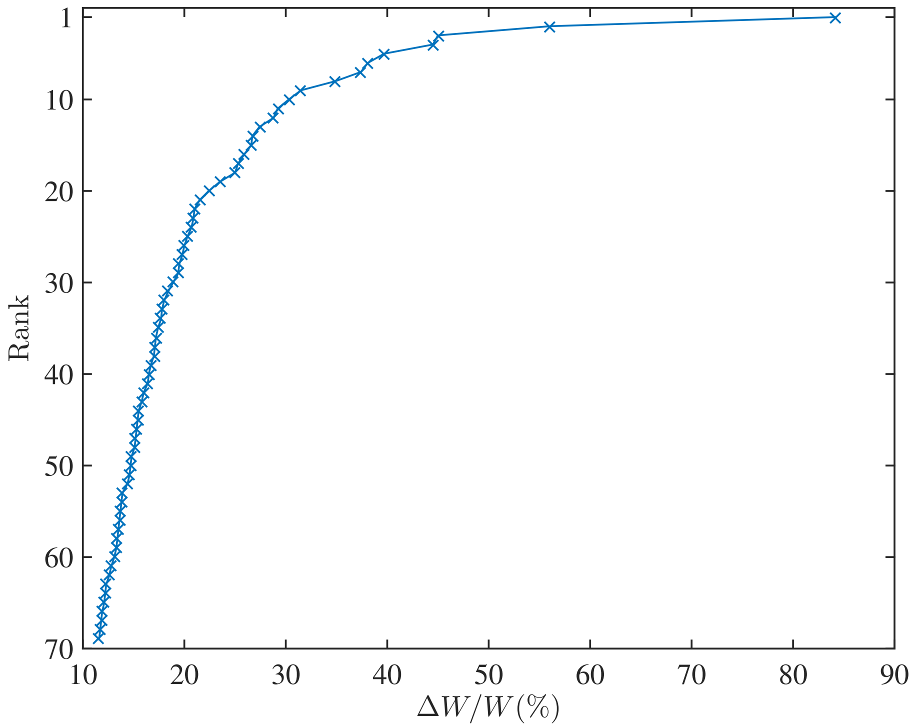

Figure 1: The rank of the 100 top performing large HFs as a function of the rate of return. A number of the HFs had the same rate of return and therefore, were equally ranked.

for the risky asset cannot be met with the cash held by a HF at a given time step, the HF requests a loan from the bank. The bank extends the loan to the HF, provided that the HF does not become more leveraged than a limit λ_{max} set by regulators. Here leverage is defined as the ratio of assets held to the HF's net wealth. In this way a constraint is imposed on the demand of the HF for the risky asset, given its wealth

$$\frac{D_t^j p_t}{W_t^j} \leq \lambda_{max}. \tag{3}$$

Consequently, the maximum demand for the risky asset is

$$D_{t,\max}^j = \lambda_{max} W_t^j / p_t. \tag{4}$$

Furthermore, we allow the HFs to take only long positions, i.e. to be active only when the asset is underpriced. We do this in order to highlight that, even with this strategy which is inherently less risky than short selling, the clustering of defaults, and thus systemic risk, is still present and is directly linked to the heterogeneity in the quality of the information available to HFs, as we explain below[5].

Therefore, the optimal demand is

$$D_t^j = \underset{D_t^j \in [0, D_{t,\max}^j]}{\arg\max} \left\{ \mathbb{E}\left[U(r_t^j) | \tilde{V}_j \right] \right\}. \tag{5}$$

To solve the optimisation problem, given that the constraint is imposed on the demand itself rather than the budget, we solve the unconstrained problem[6] and then cap the demand by equation (4), to obtain

$$D_t^j = \min\left\{ m \frac{s_j}{\alpha} W_t, D_{t,\max}^j \right\}, \tag{6}$$

where $m = V - p_t$ is the mispricing signal, given the fundamental value V. Without loss of generality in the following we set $\alpha = 1$.

HFs at each period pay management fees. Specifically, the managers receive a percentage of the wealth (standard management fee) and the realised profits (performance

[5]Similar results are obtained in the presence of short selling.
[6]In order to fascilitate analytical tractability, we assume that $V/\sigma \ll 1$.

fee) of the previous period[7], according to

$$M_t^j = \gamma \left[W_{t-1}^j + 10 \max \left(W_{t-1}^j - W_{t-2}^j, 0 \right) \right], \tag{7}$$

where $\gamma \ll 1$. The wealth of a HF evolves according to

$$W_t^j = W_{t-1}^j + (p_t - p_{t-1}) D_{t-1}^j - M_t^j. \tag{8}$$

When the wealth of a HF falls below $W_{\min} \ll W_0$, the HF is assumed to be liquidated and goes out of business. After T_r time-steps the bankrupt HF is replaced by an identical one. Finally, our model determines the price of the risky asset by a standard market clearance condition

$$D_t^{\mathrm{nt}}(p_t) + \sum_{j=1}^{K} D_t^j(p_t) = N, \tag{9}$$

where $D_t^{\mathrm{nt}}(p_t) = d^{\mathrm{nt}}/p_t$.

3 Difference in the quality of information and default statistics

Here we focus on the causes of the temporal aggregation of HF defaults. As has already been mentioned in Section 1, the default clustering can be characterised by the autocorrelation function $C(t')$, with t' being the time-lag variable. If defaults are clustered, then $C(t')$ decays such that the sum of the autocorrelation function over the lag variable diverges (**??**). Thus,

Definition 1. *Let $C(t')$ denote the autocorrelation of the time series of defaults, with t' being the lag variable. Defaults are clustered iff*

$$\sum_{t'=0}^{\infty} C(t') \approx \int_0^{\infty} C(t') dt' \to \infty. \tag{10}$$

A default of a HF is a point event occurring after a length of time (number of periods) at which the wealth of the HF drops below a threshold value $W_{\min}$. After a default

[7] In practice, HFs' management fees are paid quarterly or monthly, with a typical structure 2% of the net asset value and 20% of the net profits per annum, for the standard management fee and for the performance fee, respectively (**??**).

event an *identical* HF is reintroduced. Therefore, any default event of a specific HF, is independent of any subsequent one. Let $\{T_i^j; i = 1, 2, \ldots\}$ denote the sequence of waiting times between defaults of the jth HF. Due to the independence of defaults and the stationarity of the stochastic process governing the behaviour of the representative noise trader, T_i^j can be viewed as a sequence of independent and identically distributed random numbers in $\mathbb{N}_+$. Hence, the sequence $\{T_i^j\}$ consists an *ordinary discrete renewal process* (**?**, p. 24, 155). Intuitively, we would expect that each HF, on average, defaults at a constant rate μ^j, which, in turn, should depend only on the quality of the information about the fundamental value of the risky asset available to it, since: (a) HFs differ only in the precision of the information they have and otherwise are identical, and (b) the statistical properties of the economy as described in Section 2 are time-independent. If the assumption of a constant mean rate of default holds, the probability of $T^j = \tau$, $\tau \in \mathbb{N}_+$, is given by a geometric probability mass function (PMF)

$$P(\tau) = p^j (1 - p^j)^{\tau - 1}, \tag{11}$$

where p^j denotes the probability of default of the jth HF.

Our goal is to study the existence of systemic risk in the market and its relation with the difference in the quality of information available to each HF. A series of questions then arises: what are the statistical properties of default events on a macroscopic level? Does the market, when viewed as a whole, possess similar statistical properties to the individual HFs or is it fundamentally different? Are the defaults of the HFs correlated showing structure in the time series of defaults?

To answer these questions we study the aggregate distribution of times between defaults. To facilitate analytical treatment in the following we will treat T^j as a continuous variable. In this limit, the renewal process becomes a Poisson process, and the geometric PMF tends to an exponential probability density function (PDF)[8],

$$P(\tau; \tau \gg 1) \sim \mu^j \exp(-\mu^j \tau). \tag{12}$$

Evidently, the aggregate PDF $\tilde{P}(\tau)$ we seek to obtain is a result of the mixing of the Poisson processes governing each of the HFs. In the limit of a continuum of HFs the

[8]This limit is valid for $\tau \gg 1$ and $p^j \ll 1$ such that $\tau p^j = \mu^j$, where μ^j is the parameter of the exponential PDF [see equation (12)] (**?**).

aggregate distribution is

$$\tilde{P}(\tau) = \int_0^\infty \mu \exp(-\mu\tau)\rho(\mu) d\mu, \tag{13}$$

where $\rho(\mu)$ stands for the PDF of μ given that the quality of the information about the fundamental value s_j that each HF receives is itself a random variable in $\mathbb{R}_+$[9]. We can then show that if the PDF $\rho(\mu)$ admits a power series expansion in a neighbourhood of $\mu = 0$ then the aggregate PDF $\tilde{P}(\tau)$ decays asymptotically ($\tau \gg 1$) as a power-law.

Theorem 1. *Consider an exponential density function $P(\tau; \mu)$, parametrized by $\mu \in \mathbb{R}_+$. If μ is itself a random variable with a density function $\rho(\mu)$, and $\rho(\mu)$ in a neighbourhood of 0 can be expanded in a power series of the form $\rho(\mu) = \mu^\nu \sum_{k=0}^{n} c_k \mu^k + R_{n+1}(\mu)$, where $\nu > -1$[10], then the leading order behaviour for $\tau \to \infty$ of the aggregate probability function defined by equation (13), is $\tilde{P}(\tau) \propto \tau^{-(2+\nu+k)}$, where k is the order of the first non-zero term of the power series expansion of $\rho(\mu)$ for $\mu \to 0_+$ (exhibits a power-law tail).*

Proof. The aggregate density can be viewed as the Laplace transform $\mathcal{L}[.]$ of the function $\phi(\mu) \equiv \mu\rho(\mu)$, with respect to μ. Hence,

$$\tilde{P}(\tau) \equiv \mathcal{L}[\phi(\mu)](\tau) = \int_0^\infty \phi(\mu) \exp(-\mu\tau) d\mu. \tag{14}$$

To complete the proof we apply Watson's Lemma (?, p. 171) to the function $\phi(\mu)$, according to which the asymptotic expansion of the Laplace transform of a function $f(\mu)$ that admits a power-series expansion in a neighbourhood of 0 of the form $f(\mu) = \mu^\nu \sum_{k=0}^{n} b_k \mu^k + R_{n+1}(\mu)$, with $\nu > -1$ is

$$\mathcal{L}_\mu[f(\mu)](\tau) \sim \sum_{k=0}^{n} b_k \frac{\Gamma(\nu+k+1)}{\tau^{\nu+k+1}} + O\left(\frac{1}{\tau^{\nu+n+2}}\right). \tag{15}$$

Given that $\phi(\mu)$ for $\mu \to 0_+$ is

$$\phi(\mu) = \mu^{\nu+1} \sum_{k=0}^{n} c_k \mu^k + R_{n+1}(\mu), \tag{16}$$

[9]The distribution function of the random parameter μ is also known as the *structure* or *mixing* distribution (?).

[10]Since $\rho(\mu)$ is a PDF it must be normalisable and thus, a singularity at $\mu = 0$ must be integrable.

we conclude that

$$\tilde{P}(\tau) \propto \frac{1}{\tau^{k+\nu+2}} + O\left(\frac{1}{\tau^{k+\nu+3}}\right). \tag{17}$$

□

Corollary 1. *If $0 < k+\nu \leq 1$, then the variance of the aggregate density diverges (shows a fat tail). However, the expected value of τ remains finite.*

An important aspect of the emergent heavy-tailed statistics stemming from the heterogeneous behaviour of the HFs, is the absence of a characteristic time-scale for the occurrence of defaults (scale-free asymptotic behaviour[11]). Thus, even if each HF defaults according to a Poisson process with intensity $\mu(s)$—which has the intrinsic characteristic time-scale $1/\mu(s)$—after aggregation this property is lost due to the mixing of all the individual time-scales. Therefore, on a macroscopic level, there is no characteristic time-scale, and all time-scales, short and long, become relevant.

This characteristic becomes even more prominent if the density function $\rho(\mu)$ is such that the resulting aggregate density becomes fat-tailed, i.e. the variance of the aggregate distribution diverges. In this case extreme values of waiting times between defaults will be occasionally observed, deviating far from the mean. This will leave a particular "geometrical" imprint on the sequence of default times. Defaults occurring close together in time (short waiting times τ) will be clearly separated due to the non-negligible probability assigned to long waiting times. Consequently, defaults, macroscopically, will have a "bursty" or intermittent, character, with long quiescent periods of time without the occurrence of defaults and "violent" periods during which many defaults are observed close together in time. Hence, infinite variance of the aggregate density will result in the clustering of defaults.

In order to show this analytically, we construct a binary sequence by mapping time-steps when no default events occur to 0 and 1 otherwise. As we show below, if the variance of the aggregate distribution is infinite, then the autocorrelation function of the binary sequence generated in this manner, exhibits a power-law asymptotic behaviour with an exponent $\beta < 1$. Therefore, the autocorrelation function is non-summable and consequently, according to Definition 1 defaults are clustered.

[11]If a function $f(x)$ is a power-law, i.e. $f(x) = cx^a$, then a rescaling of the independent variable of the form $x \to bx$ leaves the functional form invariant ($f(x)$ remains a power-law). In fact, a power-law functional form is a necessary and sufficient condition for scale invariance (**?**). This scale-free behaviour of power-laws is intimately linked with concepts such as self-similarity and fractals (**?**).

Theorem 2. *Let T_i, $i \in \mathbb{N}_+$, be a sequence of times when one or more HFs default. Assume that the PDF of waiting times between defaults $\tilde{P}(\tau)$, for $\tau \to \infty$, behaves (to leading order) as $\tilde{P}(\tau) \propto \tau^{-a}$. Consider now the renewal process $S_m = \sum_{i=0}^{m} T_i$. Let $Y(t) = 1_{[0,t]}(S_m)$, where $1_A : \mathbb{R} \to \{0,1\}$ denotes the indicator function, satisfying*

$$1_A = \begin{cases} 1 & : \; x \in A \\ 0 & : \; x \notin A \end{cases}$$

If the variance of the density function $\tilde{P}(\tau)$ diverges, i.e. $2 < a \leq 3$ [see Corollary 1], then the autocorrelation function of $Y(t)$,

$$C(t') = \frac{\mathbb{E}[Y_{t_0} Y_{t_0+t'}] - \mathbb{E}[Y_{t_0}]\,\mathbb{E}[Y_{t_0+t'}]}{\sigma_Y^2},$$

where t_0, $t' \in \mathbb{R}$ and σ_Y^2 is the variance of $Y(t)$, for $t \to \infty$ decays as

$$C(t') \propto t'^{2-\alpha} \tag{18}$$

Proof. Assuming that the process defined by $Y(t)$ is ergodic we can express the autocorrelation as,

$$C(t') \propto \lim_{K \to \infty} \frac{1}{K} \sum_{t=0}^{K} Y_t Y_{t+t'}. \tag{19}$$

Obviously, in equation (19) for $Y_t Y_{t+t'}$ to be non-zero, a default must have occurred at both time t and t'[12]. The PDF $\tilde{P}(\tau)$ can be viewed as the conditional probability of observing a default at period t given that a default has occurred $t - \tau$ periods earlier. If we further define $C(0) = 1$ and $\tilde{P}(0) = 0$, the correlation function can then be expressed in terms of the aggregate density as follows:

$$C(t') = \sum_{\tau=0}^{t'} C(t' - \tau)\tilde{P}(\tau) + \delta_{t',0}, \tag{20}$$

where $\delta_{t',0}$ is the Kronecker delta. Since we are interested in the long time limit of the autocorrelation function we can treat time as a continuous variable and solve equation (20) by applying the Laplace transform $\mathcal{L}\{f(\tau)\}(s) = \int_0^\infty f(\tau)\exp(-s\tau)d\tau$, utilising also the

[12] A detailed exposition of the proof is given in Appendix A.

convolution theorem. Taking these steps we obtain

$$C(s) = \frac{1}{1 - \tilde{P}(s)}, \tag{21}$$

where $\tilde{P}(s) = \int_1^\infty \tilde{P}(\tau) \exp(-s\tau) d\tau$, since $\tilde{P}(0) = 0$. After the substitution of the Laplace transform of the aggregate density in equation (21), one can easily derive the correlation function in the Fourier space $\mathcal{F}\{C(t')\}$ by the use of the identity (**?**, p. 1129),

$$\mathcal{F}\{C(t')\} \propto C(s \to 2\pi i f) + C(s \to -2\pi i f). \tag{22}$$

to obtain (**?**),

$$\mathcal{F}\{C(t')\} \overset{f \ll 1}{\propto} \begin{cases} f^{a-3}, & 2 < a < 3 \\ |\log(f)|, & a = 3 \\ \text{const.}, & a > 3 \end{cases}. \tag{23}$$

Therefore, for $a > 3$ this power spectral density function is a constant and Y_t behaves as white noise. Consequently, if the variance of $\tilde{P}(\tau)$ is finite, then Y_t is uncorrelated for large values of t'.

Finally, inverting the Fourier transform when $2 < a \leq 3$ we find that the autocorrelation function asymptotically ($t' \gg 1$) behaves as

$$C(t') \propto t'^{2-a}, \; 2 < a \leq 3. \tag{24}$$

□

In this Section we have shown that when the default statistics of HFs are individually described by (different) Poisson processes (due to the heterogeneity in the quality of information among HFs about the fundamental value of the risky asset) we obtain a qualitatively different result after aggregation: the aggregate PDF of the waiting-times between defaults exhibits a power-law tail for long waiting-times. As shown in Theorem 1, if the relative population of very stable HFs approaches 0 sufficiently slowly (at most linearly with respect to the individual default rate μ, as $\mu \to 0$), then long waiting-times between defaults become probable, and as a result, defaults which occur closely in time will be separated by long quiescent time periods and defaults will form clusters. The latter is quantified by the non-integrability of the autocorrelation function of the sequence of default times, signifying infinite memory of the underlying stochastic process describing defaults on the aggregate level. It is worth commenting on the fact that the most stable (in terms of defaults) HFs are responsible for the appearance of a fat-tail in the aggregate PDF.

In Section 4 below we will provide evidence that the individual default rate of each HF is an increasing function of the quality of information at hand about the fundamental value of the risky asset. This is due to the fact that the demand for the risky asset is inversely proportional to the uncertainty about its fundamental value. Consequently, poorly informed HFs are on average the least leveraged and therefore, the least prone to bankruptcy due to downward fluctuations of the price of the risky asset induced by the representative noise trader.

4 Numerical simulations

In order to investigate the relevance of the conclusions we drew in the previous Section in a more realistic setting, i.e. with a finite number of HFs, we turn to numerical simulations of the model described in Section 2.

4.1 Choice of Parameters

In all simulations we consider a market with $K = 10$ HFs, each one faced with an uncertainty about the fundamental value of the risky asset σ_j^2. The inverse of the uncertainty $s_j = 1/\sigma_j^2$ defines the quality of information available to each HF, and it is uniformly distributed in the interval $[10, 100]$, unless stated otherwise. The maximum allowed leverage $\lambda_{\max}$ is set to 5. This particular value is representative of the mean leverage across HFs employing different strategies (?). The remaining parameters are chosen as follows: $\sigma^{nt} = 0.035$, $V = 1$, $N = 10^3$, $\gamma = 5 \times 10^{-4}$, $W_0 = 2$, $W_{\min} = 0.2$ and $\rho = 0.99$ (?). Bankrupt HFs are reintroduced after T_r periods, randomly chosen according to a uniform distribution in $[10, 200]$.

4.2 Results

As discussed in Section 3, one would intuitively expect each HF j to default with a constant probability p^j. At the same time, successive defaults of a HF are uncorrelated, since our model does not involve any memory of defaults taking place at earlier times, in the sense that the representative bank does not penalise a HF for going bankrupt and continues to provide credit after a HF is reintroduced and, furthermore, the risk-premium is always fixed at 0. Thus, the PMF for the waiting times between defaults

τ, was predicted to follow a geometric distribution [see equation (11)], which can be approximated by an exponential density function for $\tau \gg 1$.

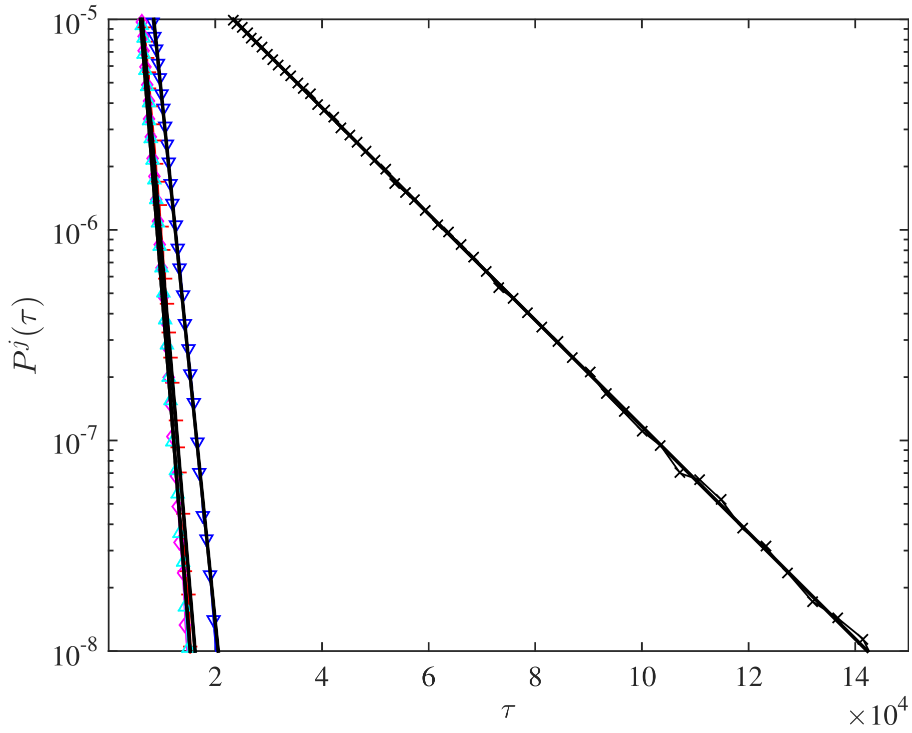

Figure 2: The PDF of waiting times between defaults τ for specific HFs, having different information quality $s_j \equiv 1/\sigma^2 = \{20, 40, 60, 80, 100\}$ (black diagonal crosses, blue downright triangles, red upright crosses, magenta diamonds and cyan upright triangles, respectively). The results were obtained simulating the model for up to 10^8 time-steps and averaging over 3×10^2 different initial conditions. In each case, we perform an exponential fit (black solid lines). Note the log-linear scale.

In Fig. 2 we show the density function $P^j(\tau)$, of waiting times τ between defaults, for a number of HFs with $s_j = 1/\sigma_j^2 = \{20, 40, \ldots, 100\}$ on a log-linear scale[13]. We observe that $P^j(\tau)$ can be well described by an exponential function for all HFs, as indicated by the fits shown with black solid lines for the various values of s_j. Therefore, we have verified that on a microscopic level

$$P^j(\tau) \sim \mu^j \exp\left[-\mu^j \tau\right] . \tag{25}$$

Moreover, as clearly shown in Fig. (2), the rate of default μ^j, corresponding to the

[13]The use of a logarithmic scale for the vertical axis transforms an exponential function to a linear one.

slope of the straight lines, is a monotonically *increasing* function of s_j. This is better illustrated in Fig. 3, where we show the mean default rate (mean number of defaults per unit time) as a function of s_j. Counter-intuitively, the probability of default increases the better the quality of the available information becomes. This can be understood as follows: A HF that knows the fundamental value with higher precision tends to become rapidly heavily leveraged. For this reason, a better informed HF is more susceptible to (downward) fluctuations of the price of the risky asset, and therefore is prone to sell (meet a margin call) in a falling market, leading eventually to its default.

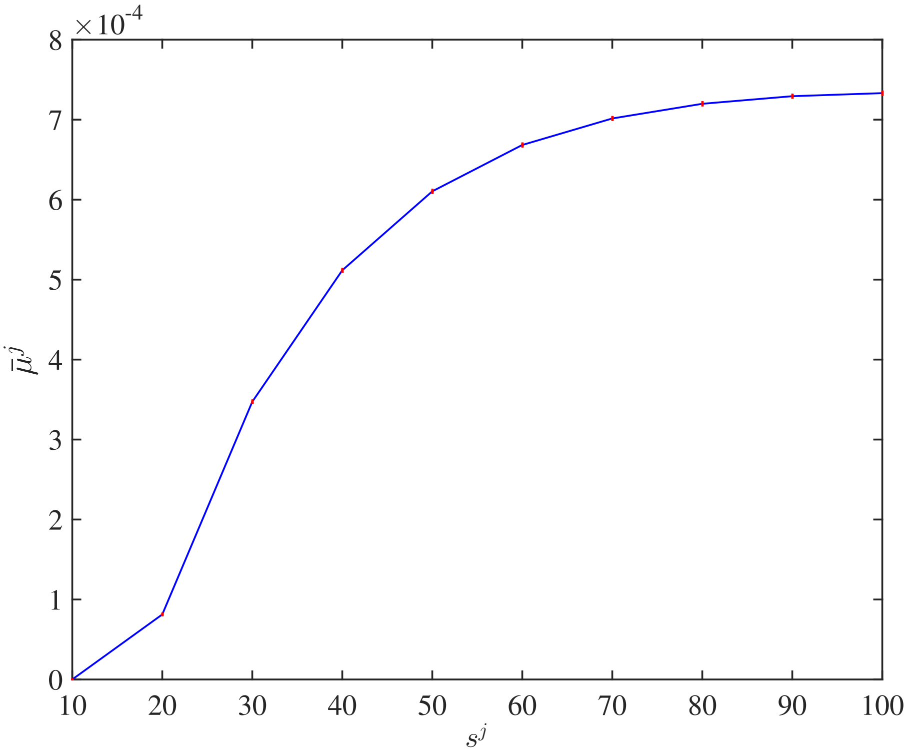

Figure 3: The mean default rate $\bar{\mu}^j$ as a function of s_j. The results were obtained simulating the model for up to 10^8 time-steps and averaging over 3×10^2 different initial conditions.

Let us now shift our attention to the aggregate statistics. In Fig. 4 we present the numerically obtained aggregate distribution using a logarithmic scale on both axes. The numerical results were obtained by averaging over 4×10^2 simulations, each with a different realisation of s_j values, sampling from a uniform distribution in the interval $[10, 100]$. We observe that for sufficiently large waiting times τ the distribution decays according to a power-law (black solid line), $\tilde{P}(\tau) \sim \tau^{-a}$ with $a \approx 7/3$. We conclude that Theorem 1 is applicable even in the case of a finite number of HFs.

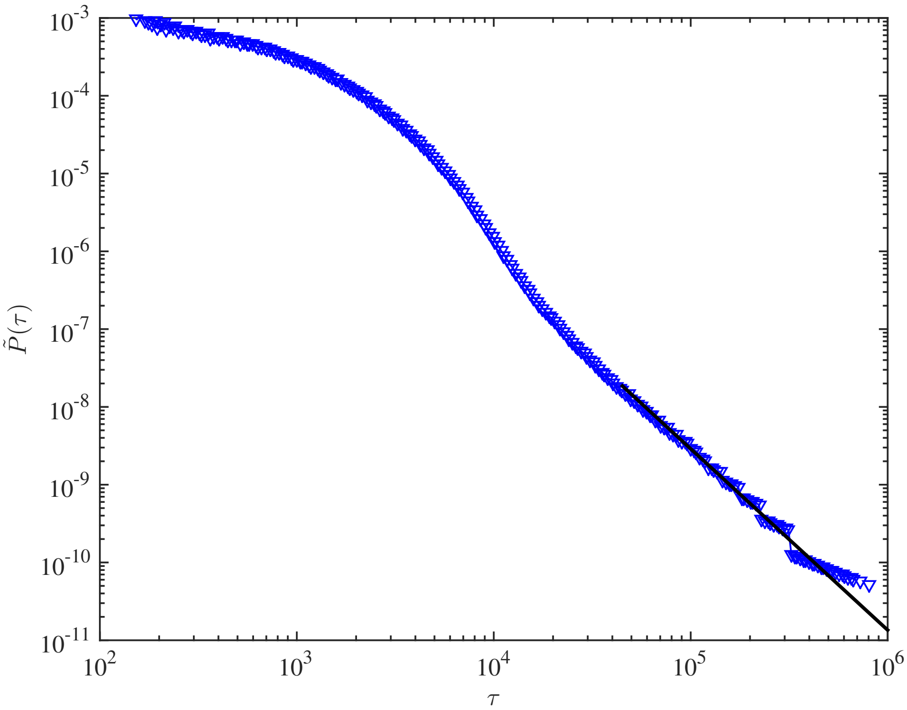

Figure 4: The aggregate PDF of waiting times between defaults (blue downright triangles) obtained on the basis of 4×10^2 simulations of the model with s_j for each HF randomly chosen according to a uniform distribution in $[10, 100]$, for up to 10^8 time-steps each. For $\tau \gg 1$, the aggregate distribution follows a power-law $\tilde{P}(\tau) \propto \tau^{-a}$, with $a \approx 7/3$ (black thick line).

Even more, the variance of the aggregate density diverges. Therefore, according to Theorem 2, the default time sequence is expected to exhibit infinite memory and thus, defaults to be clustered. Indeed, as shown in Fig. 5, the autocorrelation function decays for long lags as $C(t\prime) \sim t'^{2-7/3} = t'^{-1/3}$ (red solid line), in reasonably good agreement with the theoretical prediction of equation (24). Consequently, $\sum_{t'=0}^{\infty} C(t') \to \infty$ proving that defaults, on the aggregate level, are clustered.

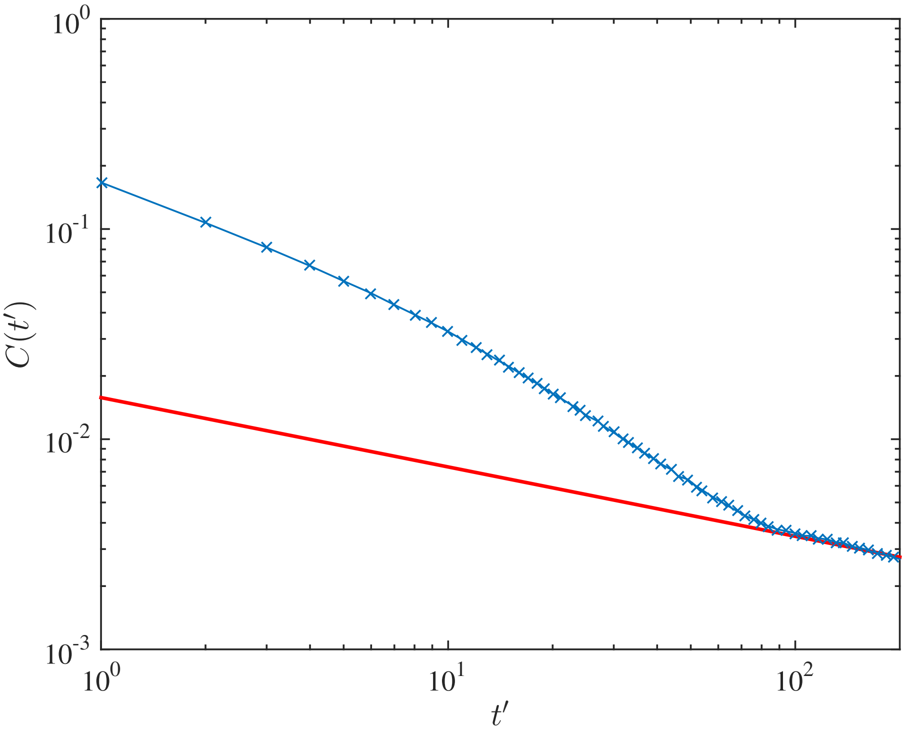

Figure 5: The autocorrelation function $C(t')$ of the binary representation of default events $Y(t)$) (blue diagonal crosses) as a function of the lag variable. The analytical prediction $C(t') \sim t'^{-1/3}$ given by equation (24) is also shown (red solid line).

4.3 Regulatory implications: Better information for all

An intriguing question relates to the relationship between the degree of heterogeneity, identified with the differnece between the extreme values of the uncertainty σ_j^2 about the fundamental value of the risky asset across all HFs, and the presence of systemic risk, i.e. clustering of defaults. Would more public (and accurate) information help suppress the clustering of defaults and therefore mitigate systemic risk?

To answer this question, we decrease the heterogeneity by improving the quality of information and compare this with the results presented in the previous section. Specifically, we assume that the s_j now lies in the interval $[10^3, 10^4]$ rather than $[10, 100]$.

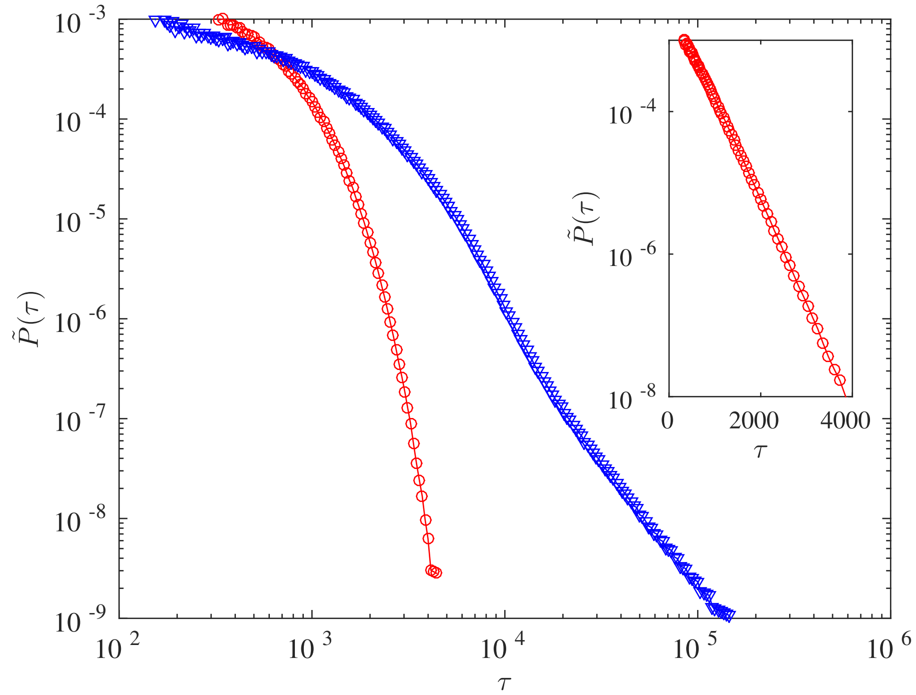

Figure 6: The aggregate PDF of waiting times between defaults assuming that $s_j \in [10, 100]$ (blue downright triangles) and $s_j \in [10^3, 10^4]$ (red circles) obtained on the basis of 3×10^2 simulations of the model for up to 10^8 periods each using double logarithmic scale. To illustrate the exponential decay of the aggregate PDF for $s_j \in [10^3, 10^4]$ we also show the corresponding aggregate density using a logarithmic scale on the vertical axis (inset).

In Fig. 6 we compare the aggregate density function obtained for s_j uniformly distributed in $[10, 100]$ (blue downright triangles)—also shown in Fig. 4—and $[10^3, 10^4]$ (red circles), using double logarithmic scale. Evidently, the power-law tail observed for $s_j \in [10, 100]$ for $\tau \gg 1$ ceases to exist when the quality of information for each HF is high, i.e. $s_j \in [10^3, 10^4]$. To better illustrate the exponential decay of the aggregate density for $s_j \in [10^3, 10^4]$ for sufficiently long waiting times between defaults we show $\tilde{P}(\tau)$ using a logarithmic scale on the vertical axis. Therefore, when all agents have a better quality of information, the aggregate density ceases to have a fat-tail and all moments of the aggregate PDF are finite.

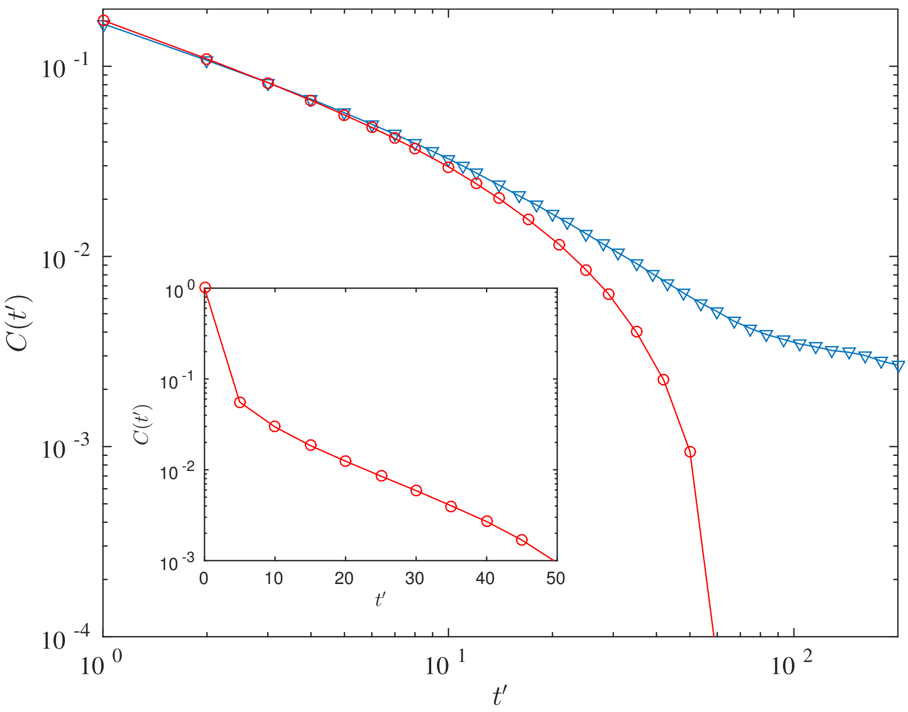

Figure 7: The autocorrelation function of the binary sequence of defaults computed numerically by averaging over 3×10^2 simulations of the model. In each simulation s_j is sampled from a uniform distribution in $[10^3, 10^4]$ (red circles) and $[10, 100]$ (blue downright triangles). The autocorrelation function corresponding to $s_j \in [10^3, 10^4]$ is also shown in the inset using a logarithmic scale on the vertical axis to help demonstrate the exponential decay in the case of better informed HFs.

It follows then from Theorem 2, the memory of the underlying stochastic process in the case of $s_j \in [10^3, 10^4]$ is finite and thus, defaults are not clustered. To confirm the theoretical prediction, which is exact in the case of a continuum of HFs, we numerically calculate the autocorrelation function of the sequence of defaults. The results are shown in Fig. 7, where the autocorrelation function corresponding to $s_j \in [10, 100]$ and $s_j \in [10^3, 10^4]$ are shown with blue downright triangles and red circles, respectively. Clearly, the decay of the autocorrelation function when HFs are better informed decays far more rapidly. In fact, as it is shown in the inset, the autocorrelation function for $s_j \in [10^3, 10^4]$

falls exponentially (short memory). Therefore, the integral of the autocorrelation function converges, and defaults are no longer clustered.

4.4 Non-normal returns and volatility clustering

The deviation from Gaussianity of the distribution of logarithmic returns in finance and the existence of infinite memory in the time series of absolute returns has been reported in numerous studies—see **?** and references therein. In the following we show that our model can replicate both of these stylized facts. In Fig. 8 we show the PDF of logarithmic returns $r = \log p_{t+1} - \log p_t$. The numerical results (black downright triangles) were fitted with a Gaussian (blue solid line). Clearly, the Normal distribution fails to describe the numerical results.

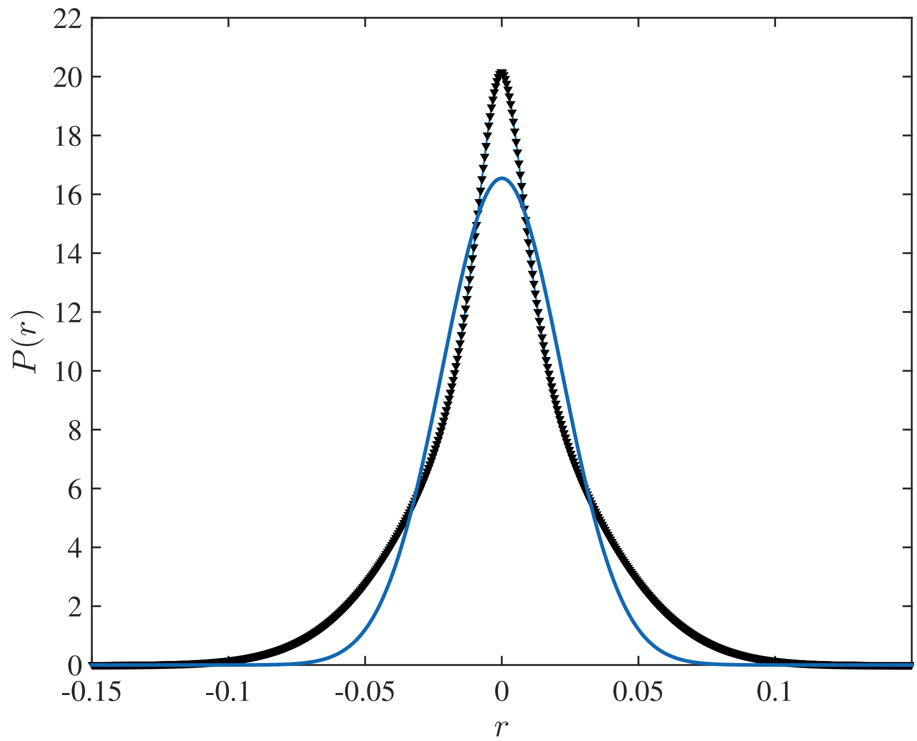

Figure 8: The PDF of the logarithmic returns computed numerically (black downright triangles). The results were obtained by averaging over 3×10^2 simulations of the model up to 10^8 periods. In each simulation s_j is sampled from a uniform distribution in $[10, 100]$. The best fit with a Gaussian distribution is also shown (blue solid line).

Finally, in Fig. 9 we present the numerically computed autocorrelation function $R(t')$ of the absolute value of logarithmic returns in double logarithmic scale. For large values of the lag variable t', the autocorrelation function behaves as a power-law, i.e. $R(t') \sim t'^{-\nu}$.

Fitting $R(t')$ for $t' \geq 300$ we find that $\nu = 0.497 \pm 3 \times 10^{-3}$. It is worth noting that the value of the exponent from empirical studies is found to be $0.2 < \nu < 0.5$, (**??**), (**?**, p. 292).

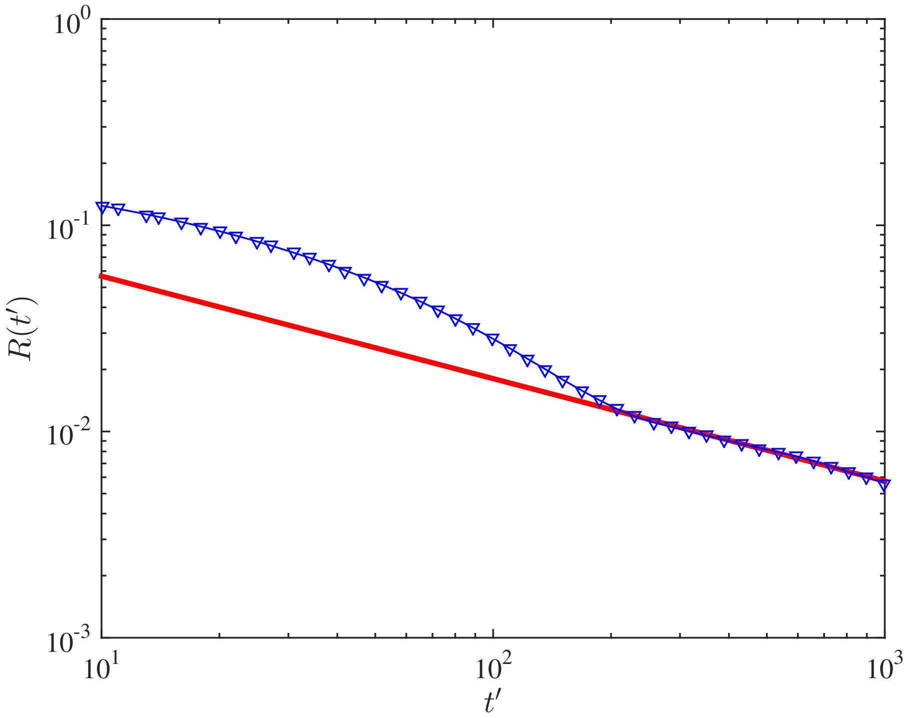

Figure 9: The autocorrelation function of the absolute logarithmic returns as a function of the lag variable (blue downright triangles). Asymptotically the function behaves as a power-law. Fitting $R(t')$ for $t' \geq 300$ (red solid line) we find that the absolute value of the exponent of the power-law is $|\nu| = 0.497 \pm 3 \times 10^{-3}$.

5 Conclusions

The rapid growth of an opaque HF industry in the last two decades constitutes a systemic risk because a synchronised failure of such investment institutions can destabilise global financial markets. Despite this widely recognised fact very few studies have been devoted to the subject. Our work makes a contribution in this direction. We relate the heterogeneity in available information among different HFs with the emergence of clustering of defaults. The economic mechanism leading to the clustering of defaults is related to the leverage cycle put forward by Geanakoplos and collaborators, according to which the presence of leverage in a market leads to the overpricing of the collateral used to back-up loans during a boom, whereas, during a recession, collateral becomes depreciated due to a synchronous de-leveraging compelled by the creditors. However, this feedback effect between collateral and leverage, as shown here, is a necessary, yet not a sufficient condition for the clustering of defaults and, in this sense, the emergence of systemic risk: the extent of heterogeneity between HFs also plays a crucial role.

Specifically, we have shown that a large difference in the quality of information available to HFs is an essential ingredient for defaults to be clustered. The mechanism for the clustering of defaults has a statistical nature. The heterogeneity among HFs, in our model realised as asymmetric information across HFs, leads to the co-existence of many time-scales characterising the occurrence of defaults. This manifests itself in the emergence of scale-free (heavy tailed) statistics on the aggregate level. We show, that this scale-free character of the aggregate survival statistics, when combined with large fluctuations of the observed waiting-times between defaults, i.e. infinite variance of the corresponding aggregate PDF, leads to the presence of infinite memory in the default time sequence. Consequently, the probability of observing a default of a HF in the future is much higher if one (or more) is observed in the recent past, and as such, defaults are clustered.

Interestingly, it is the most stable HFs responsible for the appearance of a fat-tail in the aggregate PDF, since poorly informed HFs have the lowest demand for the risky asset, and consequently, are on average the least leveraged. As a result, the HFs faced with the highest uncertainty about the fundamental value of the risky asset are the ones which are the least prone to go bankrupt due to downward fluctuations of the price of the risky asset induced by the representative noise trader.

An immediate consequence of our findings can be epitomized as follows: regulating leverage in order to mitigate the pro-cyclical expansion and contraction of credit supply, identified with the "leverage cycle", might prove inadequate. Geanakoplos correctly empasises the importance of collateral, in contrast to the conventional view, according to which the interest rate completely determines the demand and supply of credit, and thus, is the only "important variable". Heterogeneity, per se, is another destabilising factor in the economy. Therefore, future regulations should also take this into account, addressing heterogeneity of information as a source of systemic risk.

Appendix A

As already stated in Section 3, Theorem 2, assuming that the process defined by $Y(t) = 1_{[0,t]}(S_m)$ is ergodic, the auto-correlation function can be expressed as a time-average

$$C(t') \propto \lim_{K\to\infty} \frac{1}{K} \sum_{t=0}^{K} Y_t Y_{t+t'}. \tag{A.1}$$

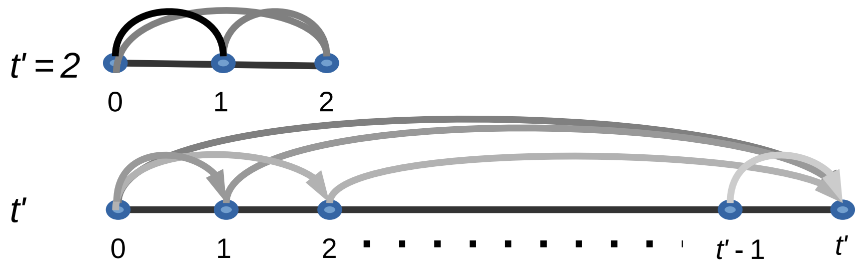

Figure 10: The probability of observing a default at t', assuming a default occurred at 0 can be expressed by $\tilde{P}(\tau = t')$.

The right-hand side (RHS) in equation (A.1) is proportional to the conditional probability of observing a default at time t', given that a default has occurred at time $t = 0$. Therefore, we can express $C(t')$ in terms of the aggregate probability $\tilde{P}(\tau = t')$ (of waiting t' time-steps for the next default to occur, given that one has just been observed).

As schematically illustrated in Fig. 10,

$$C(1) = \tilde{P}(1), \tag{A.2}$$

$$\begin{aligned} C(2) &= \tilde{P}(2) + \tilde{P}(1)\tilde{P}(1) \\ &= \tilde{P}(2) + \tilde{P}(1)C(1), \end{aligned} \tag{A.3}$$

$$\vdots$$

$$C(t') = \tilde{P}(t') + \tilde{P}(t'-1)C(1) + \dots \tilde{P}(1)C(t'-1). \tag{A.4}$$

If we further define $C(0) = 1$ and $\tilde{P}(0) = 0$, then equation (A.4) can be written more compactly as

$$C(t') = \sum_{\tau=0}^{t'} C(t'-\tau)\tilde{P}(\tau) + \delta_{t',0}, \tag{A.5}$$

where $\delta_{t',0}$ is the Kronecker delta.

We are interested only in the long time limit of the autocorrelation function. Hence, we can treat time as a continuous variable and solve equation (A.5) by applying the Laplace transform $\mathcal{L}\{f(\tau)\}(s) = \int_0^\infty f(\tau)\exp(-s\tau)d\tau$, utilising also the convolution theorem (**?**), (**?**, pp. 79-83). Taking these steps we obtain

$$C(s) = \frac{1}{1-\tilde{P}(s)}, \tag{A.6}$$

where $\tilde{P}(s) = \mathcal{L}\left\{\tilde{P}(\tau)\right\}(s) = \int_0^\infty \tilde{P}(\tau)\exp(-s\tau)d\tau$. We will assume that $\tilde{P}(\tau) \propto \tau^{-a}$ for any $\tau \in [1,\infty)$, i.e. the asymptotic power-law behaviour ($\tau \gg 1$) will be assumed to remain accurate for all values of τ. Under this assumption,

$$\tilde{P}(\tau) = \begin{cases} A\tau^{-a}, & \tau \in [1,\infty), \\ 0, & \tau \in [0,1). \end{cases}, \tag{A.7}$$

where $A = 1/\int_1^\infty \tau^{-a}d\tau = a-1$. The Laplace transform of equation (A.7) is,

$$\tilde{P}(s) = (a-1)E_a(s), \tag{A.8}$$

where $E_a(s)$ denotes the exponential integral function defined as,

$$E_a(s) = \int_1^\infty \exp(-st)\, t^{-a} dt \;/;\; \mathrm{Re}(s) > 0. \tag{A.9}$$

The inversion of the Laplace transform after the substitution of equation (A.8) in equation (A.6) is not possible analytically. However, we can easily derive the correlation function in the Fourier space (known as the power spectral density function) $\mathcal{F}\{C(t')\}(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty C(t')\cos(2\pi f)dt'$ by the use of the identity (**?**, p. 1129),

$$\mathcal{F}\{C(t')\} = \frac{1}{\sqrt{2\pi}}\left[C(s \to 2\pi i f) + C(s \to -2\pi i f)\right]. \tag{A.10}$$

relating the Fourier cosine transform $\mathcal{F}\{g(t)\}(f)$, of a function $g(t)$, to its Laplace transform $g(s)$, to obtain,

$$C(f) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-(a-1)E_a(2if\pi)} + \frac{1}{1-(a-1)E_a(-2if\pi)} \right) \tag{A.11}$$

From equation (A.11) we can readily see that as $f \to 0_+$ (equivalently $t' \to \infty$), $C(f) \to \infty$. To derive the asymptotic behaviour of $C(f)$ we expand about $f \to 0_+$ (up to linear order) using

$$E_a(2if\pi) = ai^{a+1}(2\pi)^{a-1} f^{a-1}\Gamma(-a) - \frac{2i\pi f}{a-2} + \frac{1}{a-1} + O(f^2) \tag{A.12}$$

to obtain

$$\begin{aligned} C(f) = &- \frac{i\sqrt{2\pi}(a-2)f}{4\pi^2(a-1)f^2 + (2^{a+1}\pi^a(if)^a - a(2i\pi)^a f^a)\,\Gamma(2-a)} \\ &+ \frac{i\sqrt{2\pi}(a-2)f}{4\pi^2(a-1)f^2 + (2^{a+1}\pi^a(-if)^a - a(-2i\pi)^a f^a)\,\Gamma(2-a)}. \end{aligned} \tag{A.13}$$

After some algebraic manipulation, for $f \to 0$ equation (A.13) yields

$$C(f) = Af^{a-3}, \tag{A.14}$$

where

$$A = -\frac{2^{a+\frac{1}{2}}(a-2)^2\pi^{a-\frac{3}{2}} \sin\left(\frac{\pi a}{2}\right)\Gamma(1-a)}{(a-1)}. \tag{A.15}$$

Therefore, for $2 < a < 3$ we see that the Fourier transform of the correlation function behaves as,

$$C(f) \propto f^{a-3}. \tag{A.16}$$

If $a = 3$, then the instances of the Gamma function appearing on the RHS of equation (A.13) diverge. Therefore, for $a = 3$ we need to use a different series expansion around $f \to 0_+$. Namely,

$$E_3(2\pi if) = \frac{1}{2} - 2i\pi f + \pi^2 f^2(2\log(2i\pi f) + 2\gamma - 3) + O\left(f^5\right), \tag{A.17}$$

where γ stands for the Euler's constant. The substitution of equation (A.17) into equation (A.11) leads to

$$
\begin{aligned}
C(f) = &- \mathrm{Re}\Big\{ \big[2\log(\pi f) - 2\gamma + 3 - \log(4)\big] / \big[\sqrt{2\pi}(2i\pi f \log(\pi f) \\
&+ \pi f(2i\gamma + \pi + i(\log(4) - 3)) - 2) \\
&\times (\pi(3i - 2i\gamma + \pi)f - 2i\pi f \log(2\pi f) - 2)\big]\Big\},
\end{aligned} \tag{A.18}
$$

which simplifies to

$$
\begin{aligned}
C(f) = &\Big(- 8\gamma^3\pi^2 f^2 - 2\pi^2 f(f(-6\log(\pi)(\log\left(16\pi^3\right) \\
&- 2\gamma\log\left(4\pi f^2\right)) + \left(12\gamma^2 + \pi^2\right)\log(\pi f) + 9(3 - 4\gamma)\log(2\pi f)) \\
&+ 4f\log^3(f) + 6f(2\gamma - 3 + \log(4) + 2\log(\pi))\log^2(f) \\
&+ 6f\left(\gamma\log(16) + (\log(2\pi) - 3)\log\left(4\pi^2\right)\right)\log(f) + 4f\log(2\pi)((\log(2) - 3)\log(2) \\
&+ \log(\pi)\log(4\pi)) - 4\log(2\pi f)) - 4\gamma^2\pi^2 f^2(\log(64) - 9) - 2\gamma(\pi^2 f(f\left(\pi^2 + 27 + 12\log^2(2)\right) \\
&- 4) + 4) + \pi^2 f\left(f\left(27 - \pi^2(\log(4) - 3) + \log(8)\log(16)\right) - 12\right) - 8\log(2\pi f) + 12\Big) \\
&\Big/ \Big(\sqrt{2\pi}(4\pi^2 f^2\log(\pi f)(\log(4\pi f) + 2\gamma - 3) + \pi^2 f(f(4\gamma^2 + \pi^2 + (\log(4) - 3)^2 \\
&+ 4\gamma(\log(4) - 3)) - 4) + 4)^2\Big).
\end{aligned} \tag{A.19}
$$

As $f \to 0$ we have,

$$
C(f) \sim |\log(f)| \tag{A.20}
$$

Finally, if $a > 3$, then equation (A.11) for $f \to 0$ tends to a constant, and thus, Y_t behaves as white noise. Consequently, if the variance of $\tilde{P}(\tau)$ is finite, then Y_t is for large values of t' is uncorrelated.

To summarize, the spectral density function for $f \ll 1$ is,

$$
C(f) \overset{f \ll 1}{\propto} \begin{cases} f^{a-3}, & 2 < a < 3 \\ |\log(f)|, & a = 3 \\ \text{const.}, & a > 3 \end{cases} . \tag{A.21}
$$

The inversion of the Fourier (cosine) transform in equation (A.21) yields,

$$C(t') \propto {t'}^{2-a}/;\ 2 < a \leq 3 \wedge t' \gg 1. \qquad \text{(A.22)}$$

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