Integrating Real Sector Growth and Inflation Into An Agent-Based Stock Market Dynamics*

Reiner Franke & Jaba Ghonghadze

Department of Economics, University of Kiel

FIRST AND PRELIMINARY VERSION

October 24, 2012

Abstract

This paper studies a continuous-time model of a stock market within the framework of a growing and inflationary economy, where real and financial sectors are linked through Tobin's q. Integrating a number of suitable respecifications, it succeeds in essentially re-establishing the well-known two-dimensional Lux (1995) model of an asset pricing dynamics allowing for speculative bubbles, which implicitly was set up in a stationary and non-inflationary environment.

Keywords. Tobin's q, inflationary environment, fundamentalists, chartists, market maker, speculative behavior, complex system.

1 Introduction

Formal modelling of financial markets often calls for considerable and, in many respects, unrealistic simplifications. When stressing heterogeneity of market participants, it is often the case to introduce a few type of agents who follow distinct trading rules. The number of different agents coupled with the desire to model all the complexity of their interactions very quickly may lead to insurmountable complications. This is usually the case when attempting to model the link between the real and the financial sectors. The quest for understanding this linkage is obvious with hindsight of the recent financial crisis. To build a model for a particular crisis episode is

*Email of corresponding author: jghongha@economics.uni-kiel.de

certainly worthwhile. However, this paper pursues a slightly different goal. The objective is to document the consequences for a certain well-known stock market model when augmenting it with a growing and inflationary real sector.

The point of departure of this paper is the elegant deterministic stock market model by Lux (1995) where agents are heterogeneous, fundamentalists behave in the ordinary simple way, and speculators can endogenously switch between an optimistic and a pessimistic attitude. A main result is that if herding in the nonlinear switching mechanism is sufficiently strong, locally stable bubble equilibria naturally emerge.

As in all asset pricing models of similar complexity, reference is made to an exogenously given fundamental value, the usual story being that it is given by the present value of the infinite stream of expected dividends. Unfortunately, this concept presupposes a stationary real sector. Models in their present formulation would break down if they are to be integrated with a real sector in which persistent growth and/or inflation prevails.

In such a context the 'fundamental value' has to be converted into a stationary variable, which means it has to be scaled by goods prices and a measure of economic activity. This leads us to replace the level of the stock market index with Tobin's q, the ratio of stock market valuation and the replacement value of the capital stock. For its theoretical underpinning we propose a structural framework with corporate firms in its centre who (permanently) issue equities to finance part of their fixed investment. A fundamental value of Tobin's q, designated $q^\star$, is then derived from some elementary relationships and suitable specifications. In particular, $q^\star = 1$ would make perfect sense in a steady state position.

While these concepts could be put to use in many stock market models, we subsequently return to the Lux model and adjust its formulation of the (flow) demand for equities to the new setting. Fixing the variables with which the real sector impacts on the stock market yields a two-dimensional differential equations system. Its structure is actually very similar, though not exactly equivalent, to the original Lux model. The basic dynamic properties are nevertheless maintained, which is the final upshot of the paper. The perspective for a next step (not considered in this paper) is to set up a feedback-guided (rather than New-Keynesian) version of the so-called New Macroeconomic Consensus with positive growth and inflation, where besides the real rate of interest, aggregate demand is also influenced by Tobin's q. In this way a consistent and very convenient real-financial interaction would be obtained with Tobin's q as the central link between the two sectors.

The rest of the paper is organized as follows. Section 2 introduces the stock market model. The results from the basic formal analysis are collected in Section 3. The final Section 4 concludes. All proofs and additional formal derivations can be found in the Appendix.

2 The stock market model

We begin by specifying the key relationships characterizing *production* in the economy.

2.1 Accounting identities in the real sector

The real sector of our model economy produces a single all-purpose good. It is used both for consumption and investment. If the good is not consumed immediately, it is invested.

This universal good is produced by a population of heterogeneous firms. Heterogeneity is revealed only with respect to the "mood" or the "sentiment" of the firms about the future. Here we assume only a binary representation of the opinions. Firms can be either optimistic or pessimistic.

Firms pursue a certain rule when financing their fixed investments. To characterize this rule, we need to introduce the relevant *macro* variables.[1] Let Y stand for the total output of all firms, L be the volume of employment, w denote the nominal wage rate, and δ stand for the rate of depreciation. Then the *aggregate rate of profit* r is given by

$$r = \frac{pY - wL - \delta pK}{pK}, \tag{1}$$

where p is the price of the good in monetary units, i.e. the price level, and K is the stock of fixed capital.

To proceed, we make the following two key assumptions. First, the *wage share*,

$$\omega = \frac{wL}{pY},$$

is exogenously fixed. Second, the *output-capital ratio*, $u = Y/K$, is also assumed to be given.

Given these notions, the aggregate profit rate can be expressed as follows

$$r = (1 - \omega)u - \delta. \tag{2}$$

A fraction σ_f of the profits is retained for investment and the rest is paid out to the shareholders as dividends, $0 \leq \sigma_f \leq 1$. The only alternative source of financing investment is the issuing of new shares. Here we assume a constant fraction χ of net investment pI is financed this way, $\chi \leq 1$. Note that we do not exclude the possibility of χ being negative which would correspond to the practice of corporations to buy back their equities. Together

[1] It should be noted that here we do not explicitly model the link between the decisions of single firms and the implied macro behavior.

we thus have,[2]

$$pI = \sigma_f rpK + p_e\dot{E} \tag{3}$$

with

$$p_e\dot{E} = \chi pI, \tag{4}$$

where p_e denotes the market price of the equity and $\dot{E}$ stands for the number of instantaneously issued shares. Correspondingly, E refers to the number of outstanding shares in the economy.

Denoting the growth rate by $g = I/K$, it is clear that the retention rate σ_f is determined as a residual. We obtain the relationship (no causal interpretation involved):

$$r\sigma_f = (1-\chi)g. \tag{5}$$

This equation can be interpreted as a variant of the *Cambridge equation.* In the present context it implies that if there is no equity issuing, $\chi = 0$, and there are no dividend payments, $\sigma_f = 1$, the aggregate profit rate necessarily equals the growth rate, $r = g$.

Thus, to summarize, firms finance their real investments either from their profits or by issuing shares in the financial market or by both activities. Having fixed the real sector our main goal is to build a link between it and the financial sector. Tobin's q will serve as the link between the sectors.

2.2 Tobin's q and equilibrium in the financial sector

Since equity is assumed to be the only form of the firms' outside finance, Tobin's q is given by the ratio of the market value of the firms and the replacement value of their capital stock,

$$q = \frac{p_e E}{pK}. \tag{6}$$

As an immediate consequence, from the relations $qg_e = q\hat{E} = (p_e E/pK)\times(\dot{E}/E) = p_e\dot{E}/pK = \chi pI/pK = \chi g$, we get

$$g_e = \frac{\chi g}{q}, \tag{7}$$

which shows that when Tobin's q typically moves around unity and firms do not pay out all their profits, the number of shares grow more slowly than the stock of physical capital. Furthermore, note that the situation of declining g_e is equivalent to rising q, that, *ceteris paribus*, implies growing stock price p_e.

[2]For a dynamic variable x we will use the following notation, $\dot{x} = dx/dt$ (the instantaneous increment) and $\hat{x} = \dot{x}/x$ (the instantaneous growth rate).

Equation (6) leads to the following important observation,

$$\hat{q} = \hat{p}_e + \hat{E} - \hat{p} - \hat{K}.$$

If we denote the good's price inflation rate $\hat{p}$ by π, the stock price inflation $\hat{p}_e$ by π_e, the growth rate of shares $\hat{E}$ by g_e, and recall that the capital growth rate $\hat{K}$ is given by g, the relationship reads

$$\hat{q} = \pi_e + g_e - \pi - g. \tag{8}$$

The stock market could be thought to be in equilibrium if $\hat{q} = 0$. However, such a state would imply a particular relationship between the other rates, that is

$$\pi_e = \pi + g - g_e. \tag{9}$$

From eq. (7) we may further modify eq. (9) into

$$\pi_e = \pi + g\left(1 - \frac{\chi}{q}\right). \tag{10}$$

This equation tells us that as long as $\chi < q$, taken for granted, the stock price inflation will tend to exceed the rate of goods price inflation. Most importantly, π_e will be strictly positive in such a case.

We will assume throughout that the parameters π, g, r are positive and fixed.

2.3 Participants in the financial sector

The stock market is the only type of financial market in our model. It is populated by three types of agents: speculators, who trade shares according to their sentiments; fundamentalist investors, who display a certain level of bounded rationality; and a market maker, who adjusts the market price of the shares. In the following we will give a more thorough description of these actors.

A key to understanding the functioning of our financial market is to realize that all its participants have their own beliefs and their decisions reflect them. In essence, we want to model the interaction of these beliefs. To this end we introduce a crucial variable, the *instantaneous* (or 'daily') *rate of return on holding equities*, r_e, given by the expression:

$$r_e = \frac{(1-\sigma_f)rpK}{p_e E} + \hat{p}_e = \frac{(1-\sigma_f)r}{q} + \hat{p}_e. \tag{11}$$

The first part in the expression for $\tilde{r}$ stands for the value of the dividends per share. If no portion of profits is paid out in dividends, i.e. $\sigma_f = 1$, the only rate of return on holding equities will be the instantaneous growth rate of the equity price. The latter, however, is set and controlled by the market maker. Correspondingly, the next step is to characterize the behavior of this important player.

2.3.1 Market maker

As mentioned in the Introduction, our general aim is to keep the stock market dynamics as close as possible to the model by Lux (1995) on herd behavior and bubbles. At the beginning we nevertheless face an important difference. In the Lux model, the most natural market equilibrium is the one of zero excess demand with constant equity prices, whereas in our setting we have shown above that constant values of Tobin's q imply positive stock price inflation $\pi_e > 0$. In formal terms, the market maker in the Lux model would adjust the prices following the dynamic law

$$\dot{p}_e = \beta_e \times \text{total excess demand}, \tag{12}$$

where $\beta_e > 0$ stands for the speed of adjustment. In an inflationary environment with varying stock of equities, adopted in the present work, this would have to be modified by

$$\hat{p}_e = \beta_e \times \text{total excess demand}/E, \quad \beta_e > 0. \tag{13}$$

But had the market maker followed the latter rule of price adjustments, the desired equilibrium condition with $\hat{q} = 0$ would imply the permanent positive excess demand in an equilibrium state. However, it seems questionable if the market maker would be able to avoid running out of his inventory in order to satisfy the systematically growing excess demand, even if (unrealistically) he were willing to incur a cost for replenishing it. Therefore, the dynamic law in eq. (13) has to be further modified.

We infer from this general uneasiness that the *benchmark* for the market maker's price quotes should not be, as it is usually implicitly the case, a situation of constant equity prices. He should rather see the market as acting in a basically inflationary environment and consequently use a *positive benchmark rate* π_e^m as an appropriate yardstick when reacting to excess demand. In a nutshell, in the inflationary environment the market maker would rather adopt the following rule:

$$\hat{p}_e = \pi_e^m + \beta_e \times \text{total excess demand}/E, \quad \pi_e^m > 0. \tag{14}$$

In general, we could see π_e^m as being not necessarily positive. Depending on the prevailing environment, it could be set at zero or even become negative.

The next challenge is to specify a reasonable candidate for the benchmark rate π_e^m. We postpone this until later.

Following Lux (1995), only two groups of agents form total excess demand for stocks, fundamentalists and speculators. Denoting their excess demands by d^f and d^s, respectively, eq. (14) becomes

$$\hat{p}_e = \pi_e^m + \beta_e(d^f + d^s)/E, \quad \beta_e > 0,\ \pi_e^m > 0. \tag{15}$$

2.3.2 Fundamental value and fundamentalists

Fundamentalist traders are the only long-sighted investors. They believe in the importance of Tobin's q which serves as a link between the real sector and the stock market. Apart from high values of the dividends per share (the first component in r_e) they will value their investment (or disinvestment) on the stock market with a smoother measure of the capital gains, hypothesized as π_e^f, than the rather volatile values of $\hat{p}$. Accordingly, the equity rate of return r_e^f that is relevant for them is given by

$$r_e^f = \frac{(1-\sigma_f)r}{q} + \pi_e^f, \tag{16}$$

where the important candidate for the benchmark capital gains π_e^f is the rate at which Tobin's q would hypothetically remain constant. That is, given the rate of price inflation and the two growth rates of equities and the capital stock, π_e^f is the rate that brings about $0 = \hat{q} = \pi_e^f + \hat{E} - \hat{p} - \hat{K} = \pi_e^f + g_e - \pi - g$. That is, we define

$$\pi_e^f = \pi + g - g_e. \tag{17}$$

By our assumptions, $\pi_e^f > 0$.

Recalling the reasoning that led to eq. (10), we derive

$$\pi_e^f = \pi + g\left(1 - \frac{\chi}{q}\right). \tag{18}$$

Plugging (18) into (16) yields

$$r_e^f = \pi + g - \frac{r-g}{q}. \tag{19}$$

Fundamentalists will be indifferent between holding equities and holding government bonds if r_e^f is equal to the interest rate i plus a risk premium ξ. From the perspective of the stock market, as seen by the far-sighted investors, a *fundamental value of the equity price* derives from a value of Tobin's q that equates r_e^f to $i + \xi$. Denoting this value by q^f, using (18) and (5) in (16), and solving the assumed equality, $r_e^f = i + \xi$, for $q = q^f$, we arrive at the expression

$$q^f = \frac{r-g}{i - \pi + \xi - g}. \tag{20}$$

We will refer to this q^f as an *objective* value. Correspondingly, the *objective fundamental value of the stock market index* p_e is given by

$$p_e^f = \frac{q^f pK}{E}. \tag{21}$$

Note that in a steady state position the profit rate will be equal to the real rate of interest plus risk premium, $r = i - \pi + \xi$, and that in this case the familiar benchmark value of unity will be obtained in (20). Furthermore, given the typical order of magnitude of the rates of profit and growth, the denominator in (20) will be unambiguously positive. Formally, for the analysis below we state

$$R = i - \pi + \xi - g > 0. \tag{22}$$

The emphasis placed on the *objectivity* of the fundamental values is not without a reason—fundamentalists might not measure these values with the necessary precision and, therefore, they might entertain their own *subjective* proxies, q^{fs} and p_e^{fs}, respectively. Using subjective values seems reasonable to us. Thus, the actions of fundamentalists will be based on these private perceptions.

A typical fundamentalist compares the current market value $p_e E$ of the firm sector to the subjectively perceived fundamental market value $p_e^{fs} E$ and relates them to the replacement value of the fixed capital stock. Expecting that p_e will return to p_e^{fs}, he proportionately buys (sells) if p_e^f is above (below) p_e. If β_f is a measure of the general aggressiveness of the fundamentalists and/or their number in the market and $\tilde{\beta}_e$ is an arbitrary positive constant, we have $d^f/E = \tilde{\beta}_f(p_e^{fs} - p_e)/p_e^{fs} = \tilde{\beta}_f(p_e^{fs}E/pK - p_eE/pK)/(p_e^{fs}E/pK) = \tilde{\beta}_f(q^{fs} - q)/q^{fs} = \beta_f(q^{fs} - q)$ with $\beta_f = \tilde{\beta}_f q^{fs}$. Thus, in terms of Tobin's q,

$$d^f/E = \beta_f(q^{fs} - q). \tag{23}$$

The next step is to specify excess demand of the other key players, namely, speculators.

2.3.3 Speculators

Within the group of speculative traders, the agents can switch between optimism and pessimism. If we denote by a^+ be the share of optimists and by a^- the share of pessimists (i.e., $a^+ + a^- = 1$), the percentage difference in the shares leads to the definition of the so-called *sentiment index*, $a = a^+ - a^-$. By construction, $-1 \leq a \leq 1$.

Regarding the formulation of the speculators' asset demand, we assume that per unit of time an average such trader buys (if he is optimistic) or sells (if he pessimistic) a fixed (tiny) fraction $\tilde{\beta}_s$ of the shares currently outstanding. Since with a population size of N the difference between the n^+ optimists and the n^- pessimists is easily seen to be $n^+ - n^- = aN$ (cf. Lux, 1995, pp. 886f), the demand of the speculators totals $d^s = \tilde{\beta}_s(n^+ - n^-)E = \tilde{\beta}_s aNE$ or, putting $\beta_s = N\tilde{\beta}_s$,

$$d^s/E = \beta_s a. \tag{24}$$

To close the model we will need the law of motion for the sentiment index a. Generally, it is supposed to depend on a so-called *switching index*, which we designate s_a. In the appendix we derive the following law:

$$\dot{a} = \nu[\tanh(\beta s_a) - a]. \tag{25}$$

Here $\beta > 0$ measures the "intensity of choice". The function $\tanh(x) = (\exp 2x - 1)/(\exp 2x + 1)$ is defined for all real values of x, is bounded between -1 and 1, cuts zero with positive slope, and is symmetric. If x goes to minus infinity, $\tanh(x)$ approaches -1. If x goes to plus infinity, the function values approach 1.

The switching index s_a incorporates the herding effect, the strength of which is measured by a positive coefficient ϕ_a. In addition, the propensity towards optimism increases (decreases) as these agents observe positive (negative) excess capital gains. Here we assume that the speculators choose their own positive benchmark π_e^c. The intensity of this mechanism is governed by a positive coefficient ϕ_p. Thus,

$$s_a = \phi_a a + \phi_p(\hat{p}_e - \pi_e^c). \tag{26}$$

This equation is the counterpart of eq. (9) in Lux (1995, p. 887).[3]

3 Analysis of the model

This section puts together all building blocks to derive the system of equations governing the link between the real and financial sectors. Substituting the demand (24) and (23) in (15), the price adjustments now read,

$$\hat{p}_e = \pi_e^m + \beta_e[\beta_s a + \beta_f(q^{fs} - q)]. \tag{27}$$

The variable representing the state of the stock market is Tobin's q. The law governing its motions is readily obtained from $\hat{q} = \hat{p}_e + \hat{E} - \hat{p} - \hat{K} = \hat{p}_e + g_e - \pi - g = \hat{p}_e - (\pi + g - g_e)$. It remains to substitute $\pi + g - g_e$ by π_e^f from (17) to arrive at

$$\dot{q} = q\left\{\pi_e^m - \pi_e^f + \beta_e\left[\beta_s a + \beta_f(q^{fs} - q)\right]\right\}. \tag{28}$$

Given the variables of the real sector, the stock market dynamics is completely described by the two differential equations (25) for the optimism index a and (28) for Tobin's q; clearly, π_e^m and π_e^c have to be specified in (28) and (26), respectively, for completeness.

[3]His discussion points out that the coefficient ϕ_{ap} incorporates a time dimension.

3.1 Structural equations

In summary, we obtain the following system:

$$\dot{a} = \nu \left\{ \tanh \left[\beta s_a(a,q) \right] - a \right\} \tag{29}$$

$$\dot{q} = q \left\{ \pi_e^m - \pi_e^f + \beta_e \left[\beta_s a + \beta_f (q^{fs} - q) \right] \right\}. \tag{30}$$

where

$$s_a(a,q) = \phi_a a + \phi_p \left(\pi_e^m - \pi_e^c + \beta_e \left[\beta_s a + \beta_f (q^{fs} - q) \right] \right). \tag{31}$$

In effect, we have succeeded in deriving the interactions of heterogeneous beliefs. These are represented by the interactions of the, in general, distinct, benchmark rates π_e^m, π_e^f, π_e^c, and that of the privately perceived level of the fundamental value of Tobin's q, q^{fs}. It should be stressed that none of the agents from the group of fundamentalists and chartists takes the actions of others into account. This also holds for the market maker because it suffices to him to observe only the total excess demand. Mutual ignorance of agents' choices is thus established. The next task will be to analyse the dynamic consequences of their actions.

Before switching to the dynamic analysis, three remarks stand out. First, we stress that our model is proposed to describe a situation where agents face serious limitations in observing the behavior of others. In addition, they are not learning from the observed prices and do not make inferences about the possible actions of other agents; this is, perhaps, due to high costs, high model uncertainties, and/or unreliable model outcomes. Second, it is our desire to compare the model to that of the Lux (1995). Third, we would like to try to foresee, *a priori*, the outcomes of our model and later judge these predictions from the complex system's perspective.

Following our second remark, the corresponding system from Lux (1995, eq. (10), p. 888) in original notation reads:

$$\dot{x} = 2v[\tanh(a_1 \dot{p}/v + a_2 x) - x] \cosh(a_1 \dot{p}/v + a_2 x) \tag{32}$$

$$\dot{p} = \beta \left[x T_N + T_F (p_f - p) \right]. \tag{33}$$

Note that here x has the same role as a in our model, whereas p in the Lux system corresponds to our p_e, the equity price. It is easy to show that the functional forms of a and x isoclines, on the one hand, and those of q and p isoclines, on the other, are the same. Accordingly, we have succeeded in technically replicating the Lux (1995) system.

Turning to our third remark, we note first that each group of agents entertains a specific benchmark rate of relative equity price changes which is held as a constant. In aggregate we might expect that a certain average of those rates would prevail that could in some respects affect the system

equilibrium. None of the decision-making rules, be it the one for forming an excess demand or the price adjustment rule of the market maker, taken in isolation, commands the potential existence of multiple equilibria in the system. Since *a priori* we do not have any reason to suspect the possibility of more than one equilibrium in the system, it is natural to expect that global dynamics will only be characterized by a unique equilibrium at most. This insight is interesting in light of the richness of dynamics that the Lux system possesses in reality.

We find it useful to reproduce one of the main results of Lux (1995, Proposition 2, p. 888) that regulates the system behavior. It is not difficult to show that for $\pi_e^f = \pi_e^c$ the proposition also holds for our system with very minor changes.

Proposition 3.1 (Lux, 1995)**.**

(i) For $a_2 \leq 1$ there exists a unique equilibrium $E_0 = (0, p_f)$. For $a_2 > 1$ two additional equilibria, $E_+ = (x_+, p_+)$ and $E_- = (x_-, p_-)$, emerge (again: $x_- = -x_+$ and also $p_f - p_- = p_+ - p_f$).

(ii) If $E_\pm$ exist, E_0 is always unstable.

(iii) If E_0 is a unique equilibrium (i.e. $a_2 < 1$), it may either be stable or unstable. The condition for stability is given by: $2[a_1\beta T_N + v(a_2 - 1)] - \beta T_F < 0$.

(iv) If E_0 is unique and unstable, at least one stable limit cycle exists and all trajectories of the system converge to a periodic orbit.

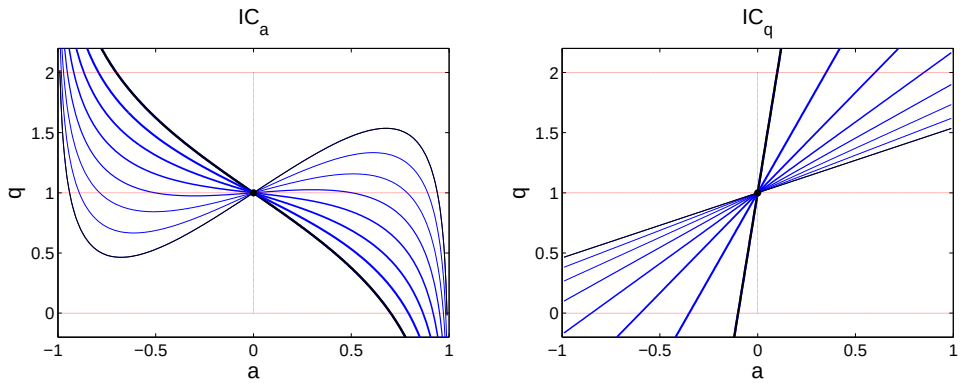
Perhaps surprisingly, the Lux (1995) system, and correspondingly, our system, may easily generate three equilibria. This is an emergent property of the system which is clearly sufficient to treat it as a *complex system.*[4] In a nutshell, our ignorant and non-learning agents in the pursuit of their selfish goals may lead to the unpredictable aggregate behavior at the system level.

In regard to our system of equations, it can be easily shown that the point $E^0 = (a^0, q^0)$ with $a^0 = 0$ and $q^0 = q^{fs} + (\pi_e^m - \pi_e^f)/(\beta_e\beta_f)$ is always an equilibrium whenever $\pi_e^f = \pi_e^c$. If in addition the condition $\pi_e^f = \pi_e^m$ holds, then the dynamic system will be anchored in the *strictly balanced equilibrium*, $(0, q^{fs})$. Interestingly, we should note that q^{fs} reflects only private and subjective perceptions about the 'fundamental' value of Tobin's q and it may deviate from what we could have called an 'objective value'. The cases when none of the benchmark rates are synchronized would lead to *almost balanced equilibrium* E^0 with $a^0 \neq 0$ and $q^0 \neq q^{fs}$.

The theoretically possible synchronization of the benchmark beliefs requires some further analysis and is currently in progress.

[4]For the key characteristics of the complex systems, and especially, in an economic context, see Durlauf (2005).

Figure 1: Phase plane and the shapes of the isoclines.



Notes: (a) The left plot shows various positions of the a-isocline, IC_a. No reference is made to the relevant parameter variations. (b) The right plot depicts the effects of increasing β_f, *ceteris paribus*, on the slope of the q-isocline, IC_q. The thickest solid line corresponds to the starting position.

3.2 Numerical analysis

As a next step we consider several numerical scenarios that enrich our understanding of the system dynamics.

We start with visualizing the possible shapes of the system isoclines in the phase plane. This is depicted in Figure 1. The left plot shows various positions of the a-isocline, IC_a. Here no reference is made to the relevant parameter variations that cause the isocline to change its shape. The most important observation is how this isocline changes the slope at the point $(0, 1)$. As seen from the figure, this slope can be positive as well as negative. The right plot depicts the effects of increasing β_f, *ceteris paribus*, on the slope of the q-isocline, IC_q. The thickest solid line corresponds to the starting position. We see that IC_q can be steep as well as flat. Most importantly, the slope of IC_q is always positive.

For completeness we provide the expressions for the isoclines:

$$\mathrm{IC}_a: \qquad q = q^{fs} + \frac{1}{\beta_e \beta_f}(\pi_e^m - \pi_e^c) + Aa - B \log \frac{1+a}{1-a}, \tag{34}$$

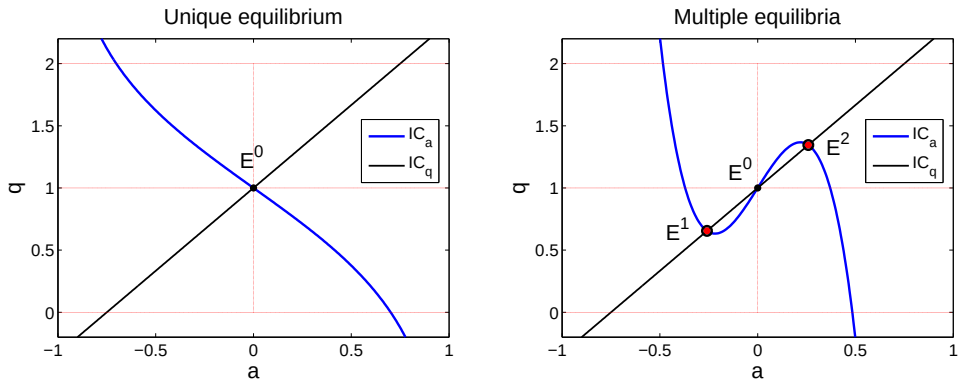
$$\mathrm{IC}_q: \qquad q = q^{fs} + \frac{1}{\beta_e \beta_f}(\pi_e^m - \pi_e^f) + \frac{\beta_s}{\beta_e \beta_f} a, \tag{35}$$

where

$$A = \frac{\phi_a + \phi_p \beta_e \beta_s}{\phi_p \beta_e \beta_f}, \tag{36}$$

$$B = \frac{1}{2\phi_p \beta \beta_e \beta_f}. \tag{37}$$

Figure 2: Phase plane and the shapes of the isoclines.



Notes: (a) The left plot depicts a scenario with a unique equilibrium, E^0. (b) The right plot visualizes a scenario with multiple equilibria. Along with the interior equilibrium E^0 we obtain two outer equilibria, E^1 and E^2.

The next point on the agenda is the visualization of possible equilibria in our system (29)–(30). Figure 2 depicts two situations. The left plot represents a scenario with a unique equilibrium, E^0. On the other hand, the right plot visualizes a scenario with multiple equilibria. Along with the *interior* equilibrium E^0 we obtain two *outer* equilibria, E^1 and E^2.

Understanding the numbers of possible equilibria we are interested in their local stability properties. To this end we can calculate the Jacobian of system (29)–(30) which has the following expression:

$$J = \begin{pmatrix} \nu\left\{\beta(\phi_a + \beta_e\beta_s\phi_p)(1-\tanh^2[\beta s_a]) - 1\right\} & -\nu\beta\beta_e\beta_f\phi_p\left(1-\tanh^2[\beta s_a]\right) \\ \beta_e\beta_s q & \pi_e^m - \pi_e^f + \beta_e\left\{\beta_s a + \beta_f(q^{fs} - 2q)\right\} \end{pmatrix} \tag{38}$$

Its sign structure is strictly fixed only in two cases,

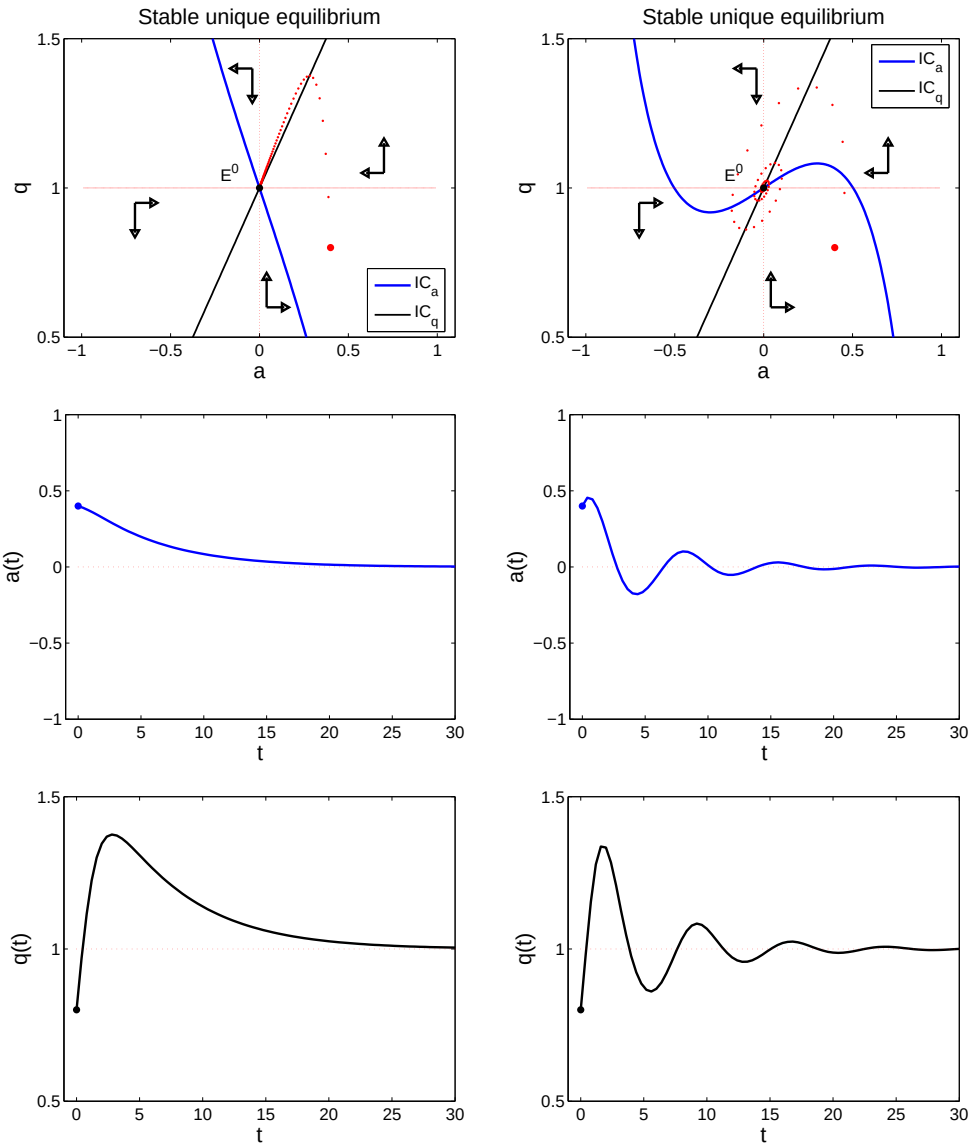
$$J = \begin{pmatrix} ? & - \\ + & ? \end{pmatrix}. \tag{39}$$

This observation leads us to resort to numerical scenarios.

The dynamics can indeed be rich. We start with a scenario when there is only one equilibrium which also is locally asymptotically stable. This situation is reproduced in Figure 3. On the one hand, converging paths towards the unique equilibrium can be monotonic. On the other hand, the trajectories approach the steady state in cycles.

In Figure 4 we switch to a scenario where the unique equilibrium is not stable anymore. Both panels in the figure are similar in that respect. Moreover, in both panels all trajectories converge towards a stable limit cycle. However, the amplitude of variations of the two limit cycles are

Figure 3: Stability of equilibria and time paths of the state variables.

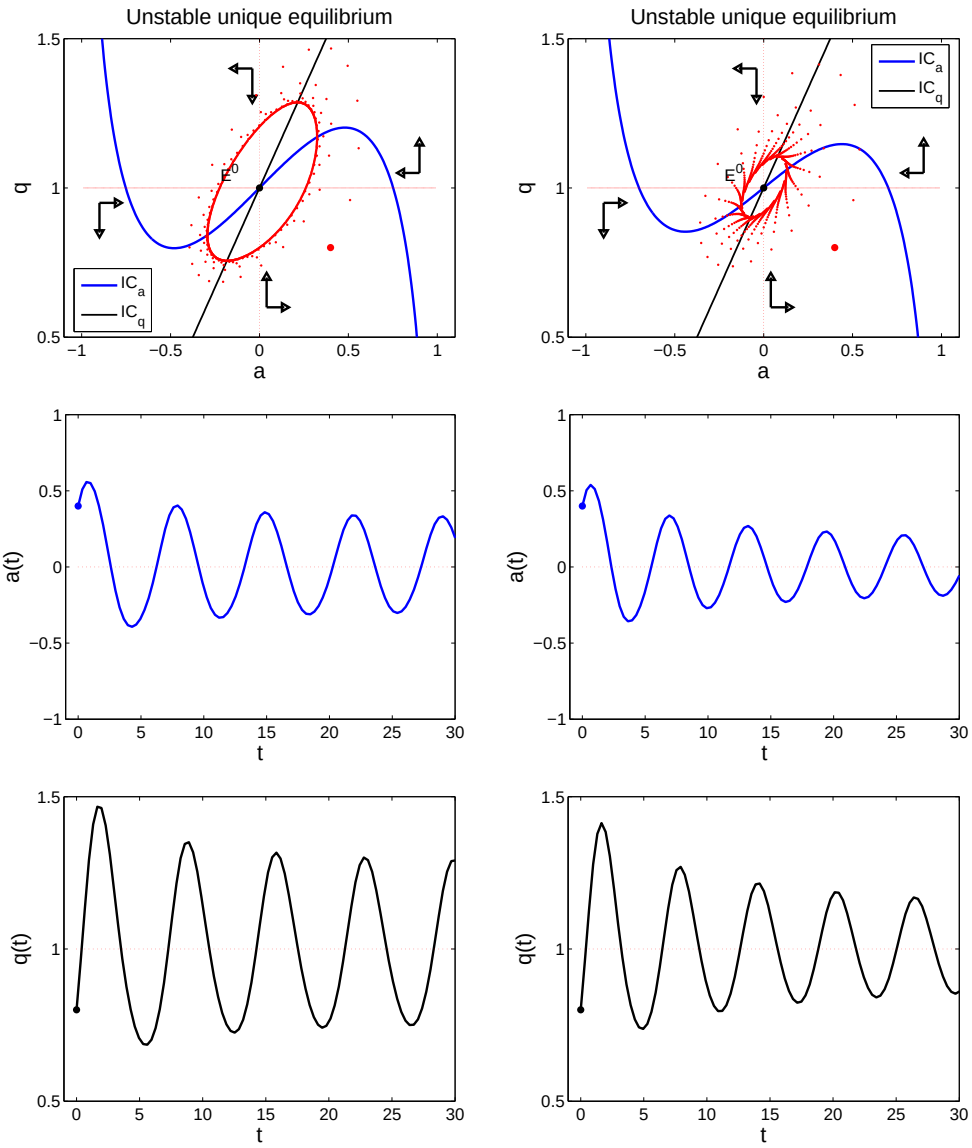


Notes: (a) The left panel depicts a scenario with a unique locally asymptotically stable equilibrium, E^0. The time paths of the state variables are monotonic. (b) The right panel also visualizes a scenario with a unique locally asymptotically stable equilibrium, E^0. However, the trajectories here are cyclic.

different. In the right panel these amplitudes are considerably smaller than in the left one. We see that the cycles with different periodicity easily emerge.

Figure 5 introduces scenarios with multiple equilibria. As soon as there are three equilibria, system behavior may become quite complex. In both

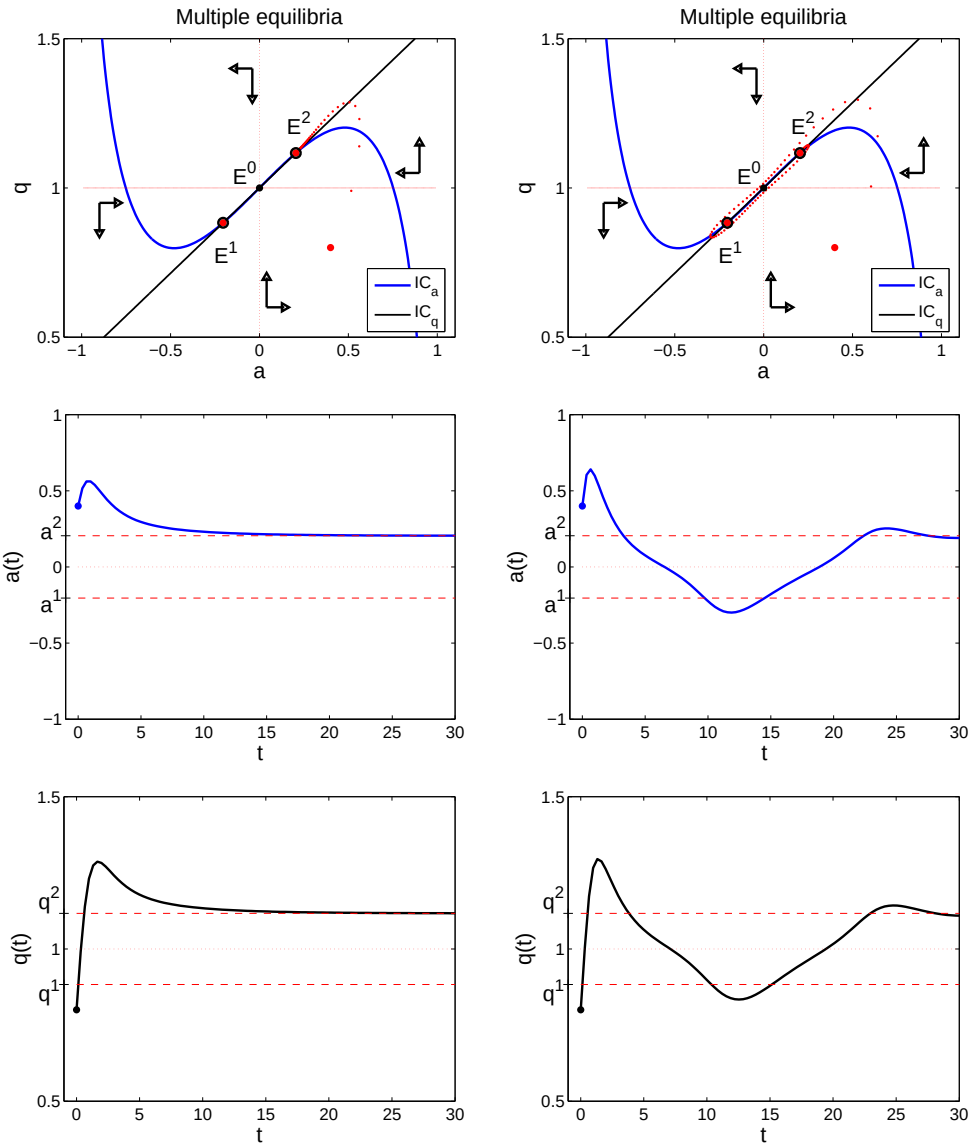
Figure 4: Stability of equilibria and time paths of the state variables.



Notes: (a) The left panel depicts a scenario with a unique unstable equilibrium, E^0. The time paths of the state variables converge towards a stable limit cycle. (b) The right panel also visualizes a scenario with a unique unstable equilibrium, E^0. However, the trajectories here converge to a "smaller" limit cycle.

panels the two outer equilibria are locally asymptotically stable. Nevertheless, convergence patterns in the two cases are quite different. The scenario in the second panel is particularly interesting. It shows that a trajectory can be very close to a lower local equilibrium, E^1, but, nevertheless, converge to the upper one, E^2, eventually.

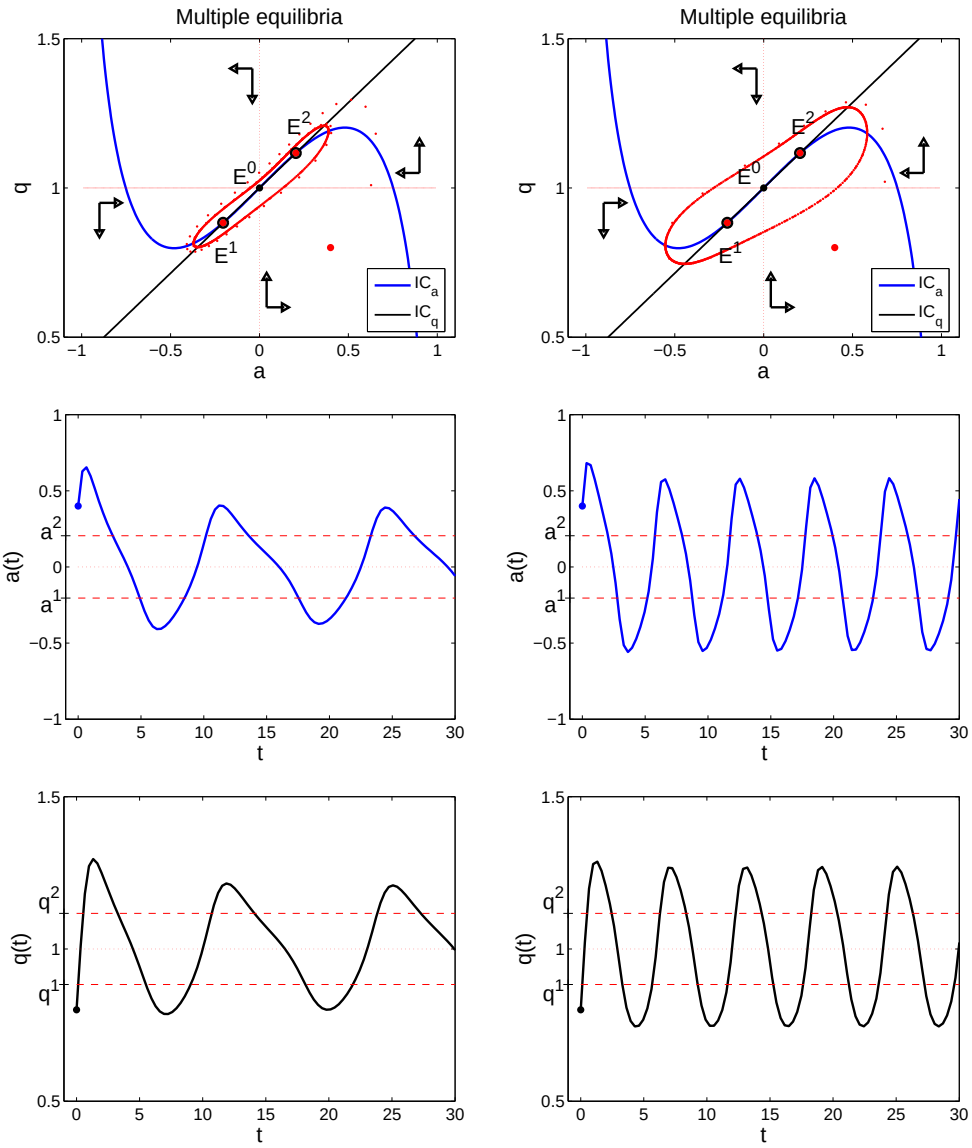
Figure 5: Stability of equilibria and time paths of the state variables.



Notes: (a) The left panel depicts a scenario with an unstable equilibrium, E^0, and two locally stable equilibria, E^1 and E^2. The time paths of the state variables converge towards a locally stable equilibrium E^2. (b) Stability properties of the equilibria in the right panel are similar to the ones in the left panel. However, the trajectory here converges to a locally stable equilibrium E^2 in a somewhat irregular way.

Another complicated case is depicted in the left panel of Figure 6. There only E^2 is locally asymptotically stable. As seen from the phase plane, there exists a limit cycle encircling all three equilibrium points. This scenario will be more thoroughly studied in our future study.

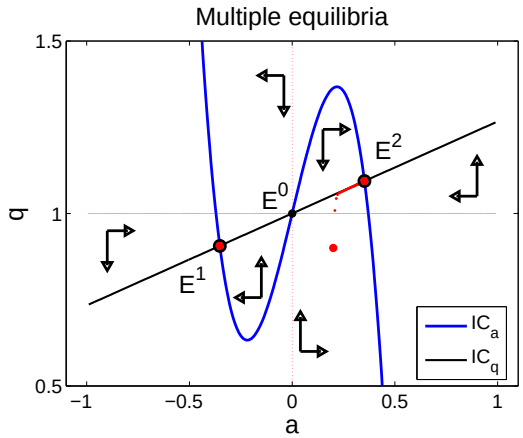
Figure 6: Stability of equilibria and time paths of the state variables.



Notes: (a) The left panel depicts a scenario with only one locally stable equilibrium, E^2. There exists a limit cycle encircling all three equilibrium points. (b) All equilibria in the right panel are unstable. However, there exists a limit cycle encircling all three equilibrium points.

We finalize our discussion by noting that when three equilibria are as in Figure 7, the phase plane contains well-defined large basins of attraction of the outer locally stable equilibria, E^1 and E^2, and an arbitrary trajectory is attracted by one of them depending on the starting position of the system.

Figure 7: Stability of equilibria.



4 Conclusions

This paper makes contributions in several respects. First of all, it studies the stock market model that is augmented by the real sector. The link between the sectors is provided by Tobin's q. Second, the enriched model is a technical replication of the well-known two-dimensional Lux (1995) model. Despite similarities, the new model contains some substantial novel elements. In particular, the economic environment is now inflationary as opposed to the implicitly stationary setting in Lux (1995). Furthermore, market maker's behavior is more flexible than before. Third, the dynamic law governing the behavior of speculative agents does not necessarily require statistical physics approach any more—it is completely reformulated. Fourth and final, our analysis highlights certain aspects of the system dynamics that have not been stressed in the earlier literature. In a nutshell, ignorant and non-learning agents in the pursuit of their selfish goals are fully capable of generating a complex system that is characterized by emergent phenomena (multiple equilibria) and unpredictable behavior at the macro level.

References

[1] Durlauf, S. (2005). 'Complexity and Empirical Economics'. *The Economic Journal 115*, 225–243.

[2] Fudenberg and Levine, 1998 ??

[3] Lux, T. (1995). 'Herd Behavior, Bubbles and Crashes'. *The Economic Journal 105*, 881–896.

Appendix

The law of motion for the sentiment index a

The changes in a are based on the so-called *switching index* s_a, which is a real function of other variables in the model that will be specified below. Its meaning is that higher values of s_a tend to increase the shares of optimists. For the moment being, let s_a be given. The law of motion for a starts out from the *discrete (binary) choice model*, which is a standard concept in agent-based modelling with time-varying group composition. It says that the population shares a^+ and a^- are directly determined by the following expressions,

$$a^+ = \frac{\exp(\beta s_a)}{\exp(\beta s_a) + \exp(-\beta s_a)}, \qquad a^- = \frac{\exp(-\beta s_a)}{\exp(\beta s_a) + \exp(-\beta s_a)}. \tag{40}$$

The (positive) parameter β is the so-called *intensity of choice*. For values of β close to zero there will be a roughly equal number of optimists and pessimists, whereas for $\beta \to \infty$, depending on the sign of s_a, almost all of the speculators tend to be either optimistic or pessimistic.

Now, for the sake of the argument, consider a sudden change in the switching index. Discrete choice has it that the population shares would immediately jump to the new levels given by (40). Our idea is that not all of the agents would be so fast in their reactions. Accordingly, we want to smooth this change and let it take place in a gradual manner. Introducing an adjustment speed $\nu > 0$, a straightforward specification of this idea in continuous time is

$$\dot{a}^+ = \nu \left[\frac{\exp(\beta s_a)}{\exp(\beta s_a) + \exp(-\beta s_a)} - a^+ \right], \tag{41}$$

$$\dot{a}^- = \nu \left[\frac{\exp(-\beta s_a)}{\exp(\beta s_a) + \exp(-\beta s_a)} - a^- \right]. \tag{42}$$

It goes without saying that the relationship $a^+ + a^- = 1$ is preserved over time. Incidentally, the adjustment equations (41)–(42) correspond to what in game theory is known as logit dynamics (Fudenberg and Levine, 1998, pp. ??). There the magnitude $1/\beta$ represents a noise level in ... ??

The differential equation for the sentiment index readily follows. In this respect it is convenient to use the function of the hyperbolic tangent, $\tanh(\cdot)$. In fact, with the hyperbolic sine, $\sinh(x) = [\exp(x) - \exp(-x)]/2$, the hyperbolic cosine, $\cosh(x) = [\exp(x) + \exp(-x)]/2$, and $\tanh = \sinh / \cosh$, equations (41)–(42) turn into

$$\dot{a} = \nu[\tanh(\beta s_a) - a]. \tag{43}$$