Institut für Makroökonomie
und Konjunkturforschung
Macroeconomic Policy Institute

Working Paper

14/2008

The J2 Status of "Chaos" in Period Macroeconomic Models

Peter Flaschel and Christian Proaño

Hans-Böckler-Straße 39
D-40476 Düsseldorf
Germany
Phone: +49-211-7778-331
IMK@boeckler.de
http://www.imk-boeckler.de

Hans Böckler Stiftung
Fakten für eine faire Arbeitswelt.

The J2 Status of "Chaos" in Period Macroeconomic Models

Peter Flaschel* Christian Proaño†

September 18, 2008

Abstract

We reconsider the issue of the (non-)equivalence of period and continuous time analysis in macroeconomic theory and its implications for the existence of chaotic dynamics in empirical macroeconomics. We start from the methodological precept that period and continuous time representations of the same macrostructure should give rise to the same quantitative outcome, i.e. in particular, that the results of period analysis should not depend on the length of the period. A simple example where this is fulfilled is given by the Solow growth model, while all chaotic dynamics in period models of dimension less than 3 are in conflict with this precept. We discuss a typical example from the recent literature, where chaos results from an asymptotically stable continuous-time macroeconomic model when this is reformulated as a discrete-time model with a long period length. ——————

Keywords: Period models, continuous time, (non-)equivalence, chaotic dynamics.

JEL CLASSIFICATION SYSTEM: E24, **E31**, **E32**.

*Department of Economics and Business Administration, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany. Email: pflaschel@wiwi.uni-bielefeld.de

†Macroeconomic Policy Institute (IMK) at the Hans-Böckler Foundation, Düsseldorf, Germany. Email: christian-proano@boeckler.de

1 Introduction

In this paper, we reconsider the issue of the (non-)equivalence of period and continuous time analysis and its implications with respect to possible chaotic dynamics in empirical macroeconomics. We start from Foley's (1975) methodological precept that period and continuous time representations of the same macrostructure should give rise to the same qualitative outcome, i.e., that the qualitative results of period analysis should not depend on the length of the period. A simple example where this is fulfilled is given by the conventional one-dimensional Solow growth model, where period and continuous analysis give qualitatively the same answer for any length of the period between zero and infinity. The assumed clustering of production and investment activities at possibly very distant points in time thus does not raise in this case the question of which period length is the most appropriate one, though it may still be asked whether the assumed type of clustering of economic activities really makes sense from an applied macroeconomic point of view if periods longer than one week or month are considered (for a detailed consideration of the role of significant lags in macrodynamics the reader is referred to Invernizzi and Medio (1991)).

We discuss in section 3 a typical example from the literature (by far not the only one), where chaos results from a asymptotically stable continuous time approach when reformulated as a "long-period" macro-model, then exhibiting a sufficient degree of locally destabilizing overshooting. As we will show, shortening the period lengths in such chaotic macro models, i.e., iterating them with a finer step size, removes on the one hand "chaos" from such model types, while it on the other hand (and at the same time) brings the model into closer contact with what happens in the data generating process of the real world.[1]

In concluding, the paper therefore proposes that continuous time modeling (or period modeling with a short period length) is the better choice to approach macrodynamical issues compared to a period model where the length of the

[1]Note in this respect again that we focus in this paper on standard period models and therefore do not yet consider, as it is done for example in Invernizzi and Medio (1991) and Medio (1991a) the role of significant delays and exponential lags in economic activity.

period remains unspecified, since it avoids the empirically uninterpretable situation of a uniform period length (with a length of one quarter, year or more) with an artificial synchronization of economic decision making. If discrete time formulations (not period analysis) are considered for macroeconomic model building they should represent averages over the day as the relevant time unit for *complete* models of the real-financial interaction on the macroeconomic level (interactions which in fact should be the relevant perspective for all *partial* macroeconomic model building). The stated dominance of continuous time modeling (or quasi-continuous modeling with a short period length) not only simplifies the stability analysis for macrodynamic model building, but also questions the relevance of period model attractors that differ radically from their continuous time analogue.

2 The J2-Status of Macrodynamic Period Analysis

Continuous vs. discrete time modeling, in macroeconomics, was discussed extensively in the 1970s and 1980s, sometimes in very confusing ways and often by means of highly sophisticated, but also by an unnecessarily complicated mathematical apparatus. There are however some statements in the literature, old and new, which suggest that period analysis in macroeconomics, i.e., discrete-time analysis where all economic agents are forced to act in a synchronized manner (with a time unit that is usually left unspecified) can be misleading from the formal as well as from the economic point of view. Foley (1975, p.310) in particular formulates the following methodological precept for the theoretical specification of macroeconomic models:

> *No substantive prediction or explanation in a well-defined macroeconomic period model should depend on the real time length of the period.*

After its intensive discussion in the 70's, this statement seems however to have become forgotten in recent times, being by far ignored in the great majority of recent analytical and numerical investigations of complex or chaotic macrodynamics. In this extent, Sims (1998) represents a prominent exception to this faulty development, based however on a different, but in spirit similar perspective on economic modeling: Sims (1998, p.318) analyzes the behavior of a variety of models with real and nominal rigidities in a continuous time formulation "[...] to avoid the need to use the uninterpretable 'one period' delays that plague the discrete time models in this literature."

In our view the core of the problem relies on to the discrepancy between the frequencies of actual data *generating* and the corresponding data *collection* processes of the great majority of macroeconomic variables. Indeed, while the actual data *generating* process at the macroeconomic level is by and large of a quasi continuous-time nature (with a less than daily frequency), the corresponding data *collection* frequency available nowadays, at least in the real markets of the economy, is on a quarterly or even yearly basis. This discrepancy is ignored in the majority of empirical mainstream macroeconomic models which, focusing on aggregate macroeconomic variables available in general at a quarterly basis, simply assume for the time intervals of the theoretical framework the same periodicity as the data collection process. This strategy, which is conditioned through the data collection technology available nowadays, can be misleading when the resulting dynamic properties of the calibrated theoretical model depend not on its intrinsic characteristics, but mainly on the length of the iteration intervals.

This issue becomes particularly clear in discrete-time dynamic models of dimensions one or two which exhibit chaotic properties, whereas in their analogous continuous time representations the occurrence of such chaotic dynamics is simply impossible from the mathematical point of view. This implies that empirically applicable period macromodels (using annualized data) should be iterated with a much finer frequency (for example with step size between "1/365 year" and "1/52 year" with respect to the actual performance of economy) in order for them to generate results that may then in general equivalent to the ones of their continuous time analogue (at least in dimensions one and two). Furthermore, models that contain expectational variables may be referring to the data collection process, yet are subject to expectational smoothing and thus should also be updated in shorter time intervals than the actually observed data.

These empirically applicable period models – which take account of the fact that macroeconomic (annualized) data are generally updated each day – will then not be able to give rise to chaotic dynamics in dimensions one and two, suggesting that the literature on such chaotic dynamics is of questionable empirical relevance (though mathematically often demanding and of interest from this point of view). To exemplify this we consider in this paper a 1D nonlinear production and real wage dynamics that has been used in a recent issue of this Journal in a period framework to generate from its parameters a period doubling route to chaos.

As a generalizing statement and conclusion, related to Foley's (1975) observation, we would conclude that the empirical relevance of macroeconomic models specified with a uniform period length across all sectors and activities and with attractors whose dynamic properties differ substantially from their continuous-time analogue should be questioned (this point in particular is addressed to all macro-approaches that derive chaotic dynamics from 1 or 2 dimensional dynamical systems, a very wide range of literature in macrodynamics, and is thus not intended to specifically criticize the 1D example here considered, since this problem is neglected by many (prominent) authors in this type of literature). Period models (and chaotic dynamics therein) thus in general depend on their continuous-time analogues (possibly – if more advanced – with some time delays) for their results, if empirically meaningful, and thus exhibit, in terms of U.S. migration policies, only a "J2 status" (dependent on a J1 visitor with work permission) in their macroeconomic implications. The next section shows by means of a recent example from the literature what how this finds formal expression in a basic one-dimensional dynamical system.

3 1D Chaotic Employment Cycles?

We start with a brief discussion of the model analyzed in Roa, Vazquez and Saura (2008) in its original discrete time formulation, which uses an unspecified period length, as it is nowadays common in the large majority of macroeconomic models.

The production of final goods is assumed to be determined by a single-input production function according to which

$$Y_t = \mu(\gamma L_t h_t)^\alpha,$$

with μ as the sector productivity, L_t as the total employment and h_t as the level of labor-enhancing technology at time t and γ as the fraction of time devoted by people for the production of final goods.

Final production is assumed to equal the next period's total demand in every period, i.e. $Y_t = D_{t+1}$. Aggregate demand for final goods in turn is assumed to equal aggregate consumption in every period, that is, $D_{t+1} = C_{t+1}$. Consumption in turn is given by

$$C_{t+1} = w_t L_t,$$

where w_t denotes the real wage and L_t the level of total employment at t. Since $Y_t = D_{t+1} = C_{t+1}$, it follows

$$Y_t = w_t L_t.$$

By equating the demand and supply expressions and solving for L_t, Roa et al. obtain the following term for the total labor demand in the economy:

$$L_t = \left(\frac{\mu(\gamma h_t)^\alpha}{w_t}\right)^{\frac{1}{1-\alpha}}. \tag{1}$$

Concerning the evolution of technical process, the stock of labor-augmenting knowledge is assumed to grow at a given rate composed of the fraction $1-\gamma$ of people devoted to the accumulation of their stock of knowledge and of a productivity index δ, that is

$$h_t = \exp(\delta(1-\gamma))h_{t-1}, \quad 0 < \delta < 1. \tag{2}$$

The fourth equation is a linear real wage Phillips curve as used for example in Goodwin's (1967) growth cycle model, namely

$$\frac{w_{t+1}}{w_t} = \exp(-a + bL_t), \quad b > a > 0. \tag{3}$$

Note that we neglect in contrast to Roa et al. natural growth, assuming a labor supply growth rate n equal to zero and normalizing the then given labor supply A to 1 in order to simplify the presentation of the dynamics slightly. The variable L is then equal to the rate of employment as in the Goodwin (1967) model.

Using the expression given by eq.(1) for t and $t+1$ delivers

$$\frac{L_{t+1}}{L_t} = \left(\left(\frac{h_{t+1}}{h_t}\right)^{\alpha} \frac{w_t}{w_{t+1}}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\exp(\delta(1-\gamma))^{\alpha}}{\exp(-a+bL_t)}\right)^{\frac{1}{1-\alpha}}. \tag{4}$$

By taking logarithms and then exponentials, Roa et al. arrive finally to the following one dimensional law of motion for the labor market dynamics

$$L_{t+1} = \exp\left(\frac{\alpha\delta(1-\gamma)+a}{1-\alpha}\right) \cdot \exp\left(\frac{-bL_t}{1-\alpha} L_t\right), \tag{5}$$

from which they generate the chaotic dynamics in employment and economic growth discussed in their article.

The continuous time reformulation of the framework by Roa et al. (2008) is given by eqs. (1)-(3)

$$L = \left(\frac{\mu(\gamma h)^{\alpha}}{w}\right)^{\frac{1}{1-\alpha}}. \tag{6}$$

$$\hat{h} = \dot{h}/h = \delta(1-\gamma) \tag{7}$$

$$\hat{\omega} = \dot{\omega}/\omega = -a + bL \tag{8}$$

In the following we measure time in years and note that the definition of a derivative like $\dot{\omega} \approx \frac{\omega_{t+\Delta t}-\omega_t}{\Delta t}$ automatically produces annualized values for the growth rates of the model. Since growth rates are thus measured in annualized form we can assume for the Phillips curve as numerical value approximately $b = 0.5$ and for δ the value 1 as a first guess (with $\alpha = 0.7, \gamma = 0.5$ and $a = 0$). This only crudely exemplifies the size we can expect for the above parameters values in the later stability investigations.

By means of conventional rules for growth rate calculations we obtain from eq.(6) together with eqs. (7) and (8) the following continuous time law of motion of the considered economy in terms of the state variable L

$$\hat{L} = \frac{\alpha\hat{h}-\hat{\omega}}{1-\alpha} = \frac{\alpha\delta(1-\gamma)+a-bL}{1-\alpha} = r - sL, \tag{9}$$

with $r = \frac{\alpha\delta(1-\gamma)+a}{1-\alpha}$, $s = \frac{b}{1-\alpha}$.

We use as state variable in this law of motion in the following however the variable $\ell = \ln(L)$ which transforms the above law of motion into the following form:

$$\dot{\ell} = r - s\exp(\ell) \tag{10}$$

Note that by defining $\ell = \ln(L)$, eq.(4) can be reformulated as

$$\Delta\ell_t = \ln\left(\frac{L_{t+1}}{L_t}\right) = \frac{1}{1-\alpha}[\alpha\delta(1-\gamma) - a + b\exp(\ell_t)] \tag{11}$$

delivering the analogous discrete time expression for the change of the variable ℓ.

The interior steady state of the dynamic law of motion described by eq.(4) is given by:[2] $\ell_o = \ln r - \ln s$ and it is of course a global attractor for all negative initial values of the variable ℓ.[3] In continuous time there is thus (of course) no way whereby complex dynamics can arise in this model type. How then do the authors obtain such a result in the discrete time analogue of the considered model? To show this we start from the (mathematically obvious) discrete time approximation, with step size Δt :

$$\ell_{t+\Delta t} - \ell_t \quad = \quad \Delta t(r - s\exp(\ell_t)), \quad \text{i.e.,} \quad \ln\left(\frac{L_{t+\Delta t}}{L_t}\right) = \Delta t(r - sL_t) \tag{12}$$

In terms of logarithms this is exactly the difference equations considered by Roa et al. (2008, p.7) in exponential form, if $\Delta t = 1$ is assumed in addition.

As the diagram illustrated in Table 1 shows, our continuous time specification delivers a valid approximation of the dynamics specified by Roa et al. in discrete time, delivering at the end (despite of our use of logarithms in the structural equations) a correct approximation in continuous time for the core dynamics of the model, namely of employment.

For a stable equilibrium point we need that there holds the condition:

$$\|1 - \Delta t\, s\exp(\ell_o)\| < 1, \quad \text{i.e.,} \quad \|1 - \Delta t\, r\| < 1 \text{ or } \Delta t < 2/r$$

[2]Since $L_o < 1$ is needed in order to run the model with less than full employment, i.e., we need the side condition $r < s$ in order to achieve this.

[3]We restrict ourselves to regimes of less than full employment here.

Table 1: "Commutative" Diagram

Discrete Time Model		Continuous Time Model
(1) $L_t = \left(\frac{\mu(\gamma h_t)^\alpha}{w_t}\right)^{\frac{1}{1-\alpha}}$	$\Longleftrightarrow$	(6) $L = \left(\frac{\mu(\gamma h)^\alpha}{w}\right)^{\frac{1}{1-\alpha}}$
(2) $h_t = \exp(\delta(1-\gamma))h_{t-1}$	$\underset{\ln(1+h)\approx\ln(h)}{\Longrightarrow}$	(7) $\ln\left(\frac{h_t}{h_{t-1}}\right) \approx \hat{h} = \delta(1-\gamma)$
(3) $\frac{w_{t+1}}{w_t} = \exp(-a+bL_t)$	$\underset{\ln(1+w)\approx\ln(w)}{\Longrightarrow}$	(8) $\ln\left(\frac{w_t}{w_{t-1}}\right) \approx \hat{\omega} = -a+bL$
(4) $\frac{L_{t+1}}{L_t} = \left(\frac{\exp(r)}{\exp(sL_t)}\right)$		$\Downarrow$
$\Uparrow \ln\left(\frac{L_{t+\Delta t}}{L_t}\right) = \ell_{t+\Delta t} - \ell_t$		$\Downarrow$
(12) $\frac{\ell_{t+\Delta t}-\ell_t}{\Delta t} = r - s\exp(\ell_t)$	$\underset{\text{proxy}}{\Longleftarrow}$	(10) $\dot{\ell} = r - s\exp(\ell)$

If this condition is replaced by $\Delta t > 2/r$ we have (locally) that the system moves away from its equilibrium point simply because we then have (in terms of deviations from the steady state $x_t = \ell_t - \ell_o$) the law of motion $x_{t+\Delta t} = (1-\Delta t r)x_t$ with $1-\Delta t r < -1$. In this case the system jumps around its equilibrium value with an increasing amplitude, generating thus spurious "chaotic" dynamics.

Returning to our continuous time variant of the model we would argue now that it represents the better approach from the applied perspective. The (annualized) output value Y is in reality changing each day on the macroeconomic level as does the stock of knowledge. The only variable where some lags in adjustment may occur is the real wage ω. However, the macroeconomic price index is also changing each day, as is the effective nominal wage level (while wage negotiations may occur somewhat clustered, but nevertheless also in a way that is scattered over the year). Hence, assuming a clustering of actual activities (not the observation of their realizations) of quarter or even yearly frequency is hard to digest from an empirical perspective.

A rough estimate of the value of r can be obtained by assuming for example $\alpha = 0.8, \delta = 1, \gamma = 0.5, a = 0$. This gives for r the value $r = 2$, i.e., the critical value that separates local stability from local instability if $\Delta t = 1$ holds.

In order to achieve instability as in Roa et al. (2008) it is therefore necessary to assume for the period length Δt that it exceeds 1. Since "1 year" is our point of reference in the continuous time dynamics we get (since the continuous time model is given in annualized terms as far as growth rates are concerned) that all activities are assumed in discrete time to cluster for example at each first day of the year and then remain inactive for the rest of the year. This is a type of behavior that we can expect to happen in population dynamics (for insects for example), but not within macroeconomics where most of indexes are changing each day. A macroeconomic model should therefore be operated with a step size much smaller than a year if it is meant to mirror the actual data generating process (which has to be distinguished carefully from the data collecting process which is not what the model is meant to explain). This will then guarantee that the stability condition will definitely hold and no period doubling route to chaos is possible.[4]

If stability gets lost by increasing the iteration step size Δt such that the above inequality becomes reversed, we can generate as in Roa et al. (2008) a period doubling route to chaos, but do achieve this by making the macroeconomy stiffer and stiffer in its totally synchronized or strictly clustered reaction patterns. As already stated such things may occur in nature due to breeding habits or in agriculture, both examples however, that are not of much relevance in a macroeconomy dominated by manufacturing and services. Only if the macroeconomy was moving as jerkily as a yearly – completely synchronized – natural reproduction mechanism chaos could be feasible. These chaotic dynamics, however, would rely on an assumption quite at odds with the actual

[4]See also Asada, Flaschel, Proaño and Groh (2007). Flaschel, Franke and Proaño (2008) apply the arguments of this paper to the 4D New Keynesian model with both staggered wages and prices, see Galí (2008, ch.6). They there provide a proof of determinacy for this model type, using a generalized Taylor principle as suggested by Galí (2008), a proof that is possibly unavailable in the 4D period version of that New Keynesian framework. This shows that our arguments can also be used to provide positive contributions in simplifying the analysis of mathematical models considerably.

dynamics of an economy at the macroeconomic level.

4 Concluding Remarks

We conclude this paper by pointing out again the importance of Foley's (1975) methodological precept for applied macroeconometric analysis, in particular when possible "chaotic dynamics" at the macroeconomic level are investigated.

Since continuous time modeling (or period modeling with a short period length) avoids the empirically counterfactual situation of a uniform period length of a length of one quarter, a year or more where an artificial synchronization of economic decision making is implied, we believe that it represents the better choice to approach macrodynamical issues. If discrete time formulations (not period analysis) are considered for macroeconomic model building they should represent averages over a short period length such as a day as the relevant time unit for *complete* models of the real-financial interaction on the macroeconomic level (interactions which in fact should be the relevant perspective for all *partial* macroeconomic model building). The stated dominance of continuous time modeling (or quasi-continuous modeling with a short period length) not only simplifies the stability analysis for macrodynamic model building, but also questions the empirical relevance of period macroeconomic models which dynamical properties differ radically from those of their continuous time analogues.

References

Asada, T., Flaschel, P., Proaño, C. and G. Groh (2007): Continuous Time, Period Analysis and Chaos from an Empirical Perspective. Working Paper 144, Center for Empirical Macroeconomics, Bielefeld University.

Flaschel, P., R. Franke and Proaño, C. (2008): On the Determinacy of New Keynesian Models with Staggered Wage and Price Setting. Macroeconomic Policy Institute (IMK) Working Paper 11/2008, Düsseldorf, Germany.

Foley, D. (1975): On Two Specifications of Asset Equilibrium in Macro-Economic Model. *Journal of Political Economy*, 83, 305 – 324.

Goodwin, R.M. (1967): A growth cycle. In: C.H. Feinstein (ed.): *Socialism, Capitalism and Economic Growth.* Cambridge, UK: Cambridge University Press, 54 – 58.

Invernizzi, S. and A. Medio (1991): On lags and chaos in economic dynamic models. *Journal of Mathematical Economics,* 29, 521-551..

Keynes, J.M. (1936): *The General Theory of Employment, Interest and Money.* New York: Macmillan.

Medio, A. (1991a): Discrete and Continuous-Time Models of Chaotic Dynamics in Economics. *Structural Change and Economic Dynamics,* 2, 99-118.

Medio, A. (1991b): Continuous-time models of chaos in economics. *Journal of Economic Behavior and Organization,* 16, 115 – 151. 521-551.

Roa, M.J., Vazquez, F.J. and D. Saura (2008): Unemployment and economic growth cycles. *Studies in Nonlinear Dynamics and Econometrics,* **12**, Nr.2, Article 6

Sims, C. (1998): Stickiness. Carnegie-Rochester Conference Series on Public Policy, 49, 317 – 356.

Publisher: Hans-Böckler-Stiftung, Hans-Böckler-Str. 39, 40476 Düsseldorf, Germany
Phone: +49-211-7778-331, IMK@boeckler.de, http://www.imk-boeckler.de

IMK Working Paper is an online publication series available at:
http://www.boeckler.de/cps/rde/xchg/hbs/hs.xls/31939.html

ISSN: 1861-2199

Hans Böckler Stiftung

Fakten für eine faire Arbeitswelt.